Sequential School Choice with Public and Private Schools*

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Abstract

Motivated by school admission systems used in, e.g., Turkey and Sweden, this paper investigates a sequential two-stage admission system with public and private schools. To perform the analysis, relevant axioms and equilibrium notions need to be tailored for the considered dynamic setting. In particular, a notion of truthfulness, referred to as straightforwardness, is introduced. In sharp contrast to classic one-stage admission systems, sequentiality leads to a trade-off between the existence of a straightforward (i.e., truthful) equilibrium and non-wastefulness. Given this insight, we identify the unique set of rules for two-stage admission systems that guarantees the existence of a straightforward equilibrium and, at the same time, reduces the number of wasted school seats. Several existing admission systems are also theoretically analyzed within our general framework and empirically evaluated using school choice data from Sweden. The latter analysis allows us to quantify various trade-offs in sequential admission systems.

Keywords: market design, sequential school choice, private schools, public schools, straightforward SPNE, non-wastefulness.
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1 Introduction

In most countries, both private and public schools are an integral part of the education system. In the OECD countries, for example, 18 percent of all 15-year-old students are admitted to privately managed schools. In fact, among the OECD countries and their 14 partner economies, only Azerbaijan, Romania and the Russian Federation do not have any private schools (OECD, 2012). In many of these countries, school seats are sequentially assigned to students in two or more stages. The main difference between one-stage and multi-stage admission systems is that the latter type of system must include a set of rules that determines, e.g, the subset of schools that are available in each stage, the set of students that are allowed to participate in each stage, and a stage-dependent mechanism for determining school assignments. In, for example, Boston[1] and New York City[2], the admissions to exam schools and regular public schools are conducted separately and sequentially from each other, and students are allowed to select their most preferred school from the two separate admissions. In the context of high school admissions to private and public high schools in Turkey, the order in which public and private schools admit students and whether a student can vie for both types of schools without commitment to enroll has changed in recent years. A third example is the municipality of Botkyrka, which is part of the Greater Stockholm Region in Sweden, where a centralized public school assignment follows the admissions to private schools.[3] A final example is college admissions in China which are based on a tiered-admissions system where assignments to prestigious top-tier colleges are followed by assignments to lower-tier colleges (Chen and Kesten, 2017).

The pioneering work by Abdulkadiroğlu and Sönmez (2003) initiated a large literature on school choice. The vast majority of the papers in this literature assume that students are assigned to public schools in a single, centralized admission[4] stage and aim to design matching mechanisms based on desideratum represented by formal axioms. In particular, much attention has been directed towards the trade-off between stability and efficiency while maintaining good incentive properties (see, e.g., Abdulkadiroğlu et al., 2009; Kesten, 2010). The main characteristic that distinguishes the considered framework from most of the existing literature is that our framework focuses on multi-stage admission systems that may feature any possible combination of centralized and decentralized admissions, which is a prevalent feature of almost all real-life student admissions.

When students seek assignment through sequential admissions, presumed properties possessed under static admission systems may be compromised since students need to make multiple decisions over time, which inevitably leads to spillovers across different types of schools, e.g., a student admitted both at a public and a private school may subsequently vacate her seat at either

[1] See www.bostonpublicschools.org/Page/7080 and www.bostonpublicschools.org/Page/6594.

[2] For more information, see Abdulkadiroğlu et al. (2009).

[3] For more information, see Kessel and Olme (2018).

[4] Very few papers exclusively model decentralized college admissions, c.f. Chade et al. (2014), Che and Koh (2016) and Hafalir et al. (2018).

school leading to a vacancy, which can potentially jeopardize the stability of the assignments obtained through a one-shot admissions system. A central question then concerns the trade-off between welfare and incentive issues. In particular, this paper addresses the following question: "how should a multi-stage system be ideally organized in terms of timing, participation, and assignment rules across and within stages in light of such trade-offs?"

The high school admissions system that determines the placements of over a million students in Turkey has recently undergone a series of reforms and remains a source of much controversy. In the system that was in use in 2014, students who were admitted to public schools in a centralized first stage could go on to be admitted to private schools in a decentralized second stage before eventually deciding on their actual assignments. This "admission without commitment" feature of the system gave rise to tens of thousands of seats of both types of schools to be subsequently vacated. In the aftermath of the epic number of vacancies, the ministry of education made a failed attempt to address the demand for vacated seats and coordinate the subsequent vacancy chains via multiple reassignment rounds throughout summer 2014. The highly chaotic process continued well after the academic year started.[5] The overwhelming public dismay and administrative burden with this system lead the ministry to implement a major reform through a new system the following year. Many believed that the problem with the old system had to do with the organization of the timing of admissions for the two types of schools. In particular, it was decided that private schools should run admissions before public schools in the new system of 2015. A second, more subtle change concerned the "rules of entry" into the second stage. Students who were admitted into a private school in the first stage were no longer permitted to apply to public schools in the second stage. The reforms successfully mitigated the problems associated with seat vacations observed in the previous year. This is indeed confirmed by our analysis which shows that the unique equilibrium outcome of the new system is at least as good as the truth-telling equilibrium outcome of the old system. Despite improved welfare, however, a new type of problem, one of incentives, surfaced in the new system. Families felt under heavy pressure to broadly choose between private and public schooling without actually knowing what specific assignment the admissions stage they have chosen to participate in would lead to.[6]

The admissions systems used in many Swedish municipalities bear interesting similarities to the old and new Turkish systems. In Stockholm, for example, the public and private school admissions are entirely decentralized and independently administered with little coordination in the absence of any central oversight of an entity such as the Ministry of Education in Turkey. An epidemic problem that plagued the city's education system has been due to the fact that a student can freely enroll at several schools (possibly of the same type) before eventually decid-

[5] An overall five supplementary rounds took place that year in which 13,398, 15,694, 39,037, 13,130, and 18,014 students were assigned and reassigned, respectively.

[6] A letter from a parent to the President of the Republic of Turkey echoes this concern: "The current system forces us to make a decision between public and private schools in the beginning... Hence, many parents are forming a line before private schools fearing that they would not be admitted to any public school." (Haberturk newspaper, July 2015).

ing on her actual assignment. Much like the old Turkish system, unclaimed seats have lead to severe waste and coordination problems, which has been a source of much controversy.[7] In the Swedish municipality Botkyrka, on the other hand, legislation makes it possible for private and public schools to coordinate their admissions through the municipality. Specifically, students submit joint preferences by ranking a private school together with other public schooling options. After the private schools process the applications of those who listed them as *first* choice, the unassigned private school applicants (together with those who only listed public schools) are placed to public schools based on the remaining portion of their preferences through a central assignment. The manner in which preference submissions are constrained in Botkyrka leads to an incentive problem similar to that in the new system of Turkey: if a student's true *first* choice is not a private school, by listing only public schools, she needs to forego any admission chances to a private school; or if she still wishes to apply to a private school that is not her true *first* choice, then she may miss out on a more preferred public school that would otherwise admit her. It turns out that the set of equilibria of the Botkyrka system is essentially the same as those of the new Turkish system.

Of particular interest for our analysis are systems in which only public schools are considered in the first admission stage and only private schools are considered in the second admission stage or vice versa as in, e.g., Turkey and some municipalities in Sweden. Similar to the existing literature, the point of departure is a set of desirable axioms. From the previous literature, it is known that some of these axioms are considered to be of particular importance (see, e.g., Abdulkadiroğlu and Sönmez, 2013, for an overview). One such property is non-wastefulness which means that after the matching mechanism has assigned the students to schools, there should exist no student who prefers a school with an empty seat to her assigned school. Another such property is that the matching mechanism should be designed in such fashion that it is impossible for students to gain by strategic misrepresentation of their preferences. Even if most of the considered axioms naturally can be extended to the multi-stage setting (e.g., non-wastefulness), it is more complicated to define a multi-stage notion of truthfulness. To encompass for the latter, we define a sequential notion of truthfulness called "straightforwardness". This notion means that the students report their true preferences over the (relevant) schools in each round of the sequential admission process. The notion also captures the idea of sequential rationality as it implies that students aim to improve their current situation in each admission round.

In one-stage admission systems, mechanisms that always select a non-wasteful matching where students, in addition, have incentives to truthfully report their preferences are known to exist. A prominent example of such a mechanism is the student-proposing deferred acceptance algorithm (Abdulkadiroğlu and Sönmez, 2003; Gale and Shapley, 1962). It is, however, more demanding to design a matching mechanism that satisfies these two specific properties in multi-stage admission systems due to the dynamic nature of sequential systems. That is, multi-stage

[7]This problem has, e.g., been discussed in an official proposal that was submitted to the Swedish parliament on October 6, 2015 (Motion 16:2156, 2015).

systems assign students in each stage and allow students to adopt more complex strategies. As a consequence, school seats may be wasted in each stage of the admission process, and the complex structure of the strategy sets makes it more demanding to exclude the possibility of successful manipulation. Our results also show that there is a trade-off between the existence of a straightforward Subgame Perfect Nash Equilibrium (SPNE) and non-wastefulness. In fact, the complex nature of sequential admission systems leads to that very specific admission rules, related to when students are allowed to participate in specific rounds and whether or not they can "keep" their assignment from the first stage of the admission process when participating in the second stage, are required to guarantee the existence of a straightforward SPNE. Our main theoretical result (Theorem 1) is there is a unique set of rules for sequential assignment systems that guarantees the existence of a straightforward SPNE and at the same time minimizes the waste of school seats. These results enable us to propose a new system for two-stage school admissions.

To quantify the theoretical results, the 2015 admission data from the Swedish municipality Botkyrka is analyzed. This data is ideal for our purposes since all applications are handled in a centralized application system even if admissions to private schools precede admissions to public schools. Consequently, the data contains information about how students rank public schools *in relation* to private schools and we are not aware of any other data set with this specific property. This unique property of the data set also enables us to empirically analyze the theoretical properties of several existing two-stage admission systems, e.g., the new and the old Turkish system, using the Botkyrka admission data. Our empirical analysis quantifies the cost of inducing truthfulness in a sequential assignment system. Our main ïňĄnding from this analysis is that the waste of school seats can be dramatically reduced by organizing the two-stage admission system in such a fashion that students in the ïňĄrst stage are allowed to apply only to either private or public schools depending on which of the two sets of schools that is expected to have the fewest number of applicants.

Our theoretical and empirical results suggest a way to minimize waste while preserving truthful incentives. Importantly, the proposed approach incentivizes private schools to voluntarily join centralized assignment, which in turn facilitates convergence to a uniïňĄed centralized assignment system that can readily avoid the trade-offs faced in sequential assignments.

1.1 Previous Literature

As described above, this paper deviates from the vast majority of the papers in the existing school choice literature as it considers a general framework for analysing sequential admission systems. There are very few papers that deal with multiple school admissions systems when viewed from a single lens. The three contemporaneous works that are most closely related to us are Ekmekci and Yenmez (2014), Manjunath and Turhan (2016), and Dur and Kesten (2018).

Ekmekci and Yenmez (2014) study sequential assignments where the centralized admission for district schools precedes individual admissions for charter schools. Their main focus is on schools' incentives to be part of a centralized system as opposed to remaining within the sequential system, and they show that such centralized admission is never incentive compatible for charter schools. A charter school is better off running its independent admissions after the centralized admissions. To remedy this problem, they propose a mechanism with virtual charter schools and accordingly show that an equilibrium then can be sustained if all schools participate in the centralized clearinghouse.

Manjunath and Turhan (2016) investigate a school district where groups of schools run their admissions processes in parallel[8] and independently from each other. They show that the resulting school assignment is often inefficient, and offer a way to Pareto improve upon these assignments by iteratively rematching students. Dur and Kesten (2018) underline the conflict between efficiency and incentives in sequential systems where students are forced to choose which stage of admissions to participate in and argue that unified admissions leads to superior welfare and incentive properties.

Differently than these approaches, our focus is on private vs. public school admissions. Given the fact that there are institutional barriers that categorically preclude unification of admissions processes for both type of schools, admissions are inherently separate. As such we do not seek to unify the admissions processes across different stages into a single centralized round, but rather ask how should the "right" sequential system be operationalized in the face of such constraints. Parallel admission systems and limited sequential systems such as those in Dur and Kesten (2018) whereby a student can be assigned in at most one stage, are special cases of our general model. To our knowledge, a model with this level of generality has not been considered before. Such a framework proves useful in assessing the trade-offs both theoretically and empirically when the planner is unconstrained in the organization of the admission system and her choice of the underlying sets of rules.

One of the early papers that addressed sequentiality in a matching framework is due to Alcalde et al. (1998), where a two-stage mechanism is proposed for a two-sided job market. Their mechanism implements the set of stable matchings in SPNE. Similarly, in many-to-one matching markets, Alcalde and Romero-Medina (2000) show that the set of stable matchings can be obtained as an equilibrium when agents play in a sequential manner.[9]

Abdulkadiroğlu et al. (2009) provide an informal discussion of sequential school assignments in New York City and argue that the current multi-round assignment plan may result in unstable student assignments. Westkamp (2013) studies the German college admissions system which operates through a combination of the Boston mechanism and the college proposing deferred acceptance algorithm. He demonstrates that the set of SPNE is characterized by the set of stable

[8] Anno and Kurino (2016) similarly study parallel admissions.

[9] As for many-to-many matching markets, Echenique and Oviedo (2006), Romero-Medina and Triossi (2014) and Sotomayor (2004), provide similar characterizations.

matchings.

In recent works, Doğan and Yenmez (2018a) and Doğan and Yenmez (2018b) analyze recent developments in the Chicago school system in which there are unified and divided enrollment systems for different types of schools. In the former paper, they compare the welfare of students under unified and divided enrollment systems when students are not strategic. In the latter paper, they allow students to be strategic in a two-stage game in which the same set of schools are available under both stages. Finally, Haeringer and Iehle (2017) analyze the multi-stage college admission system in France.

1.2 Outline of the paper

The remaining part of the paper is outlined as follows. Section 2 introduces the school choice problem with private and public schools together with some important definitions and axioms. The sequential student assignment game is stated in Section 3. This section also includes a formal definition of the notion of straightforwardness. The general theoretical results are provided in Section 4 whereas the theoretical results related to country specific admission systems are stated in Section 5. The empirical analysis is contained in Section 6. Section 7 concludes the paper. The appendices contain a more formal description of the extensive form student assignment game, the proofs of the theoretical results as well as a more detailed analysis of the decentralized Turkish private school admission game.

2 The School Choice Problem with Private and Public Schools

This section introduces the basic ingredients of the school choice problem with private and public schools together with a number of important concepts, axioms and definitions.

2.1 The Basic Model

A school choice problem with private and public schools consists of a set of private schools, a set of public schools, the capacities of the schools, and a set of students. Students are assumed to have strict preferences over the schools and the option of remaining unassigned. Schools are endowed with strict priority orderings over the students and the option of leaving a school seat unfilled. Formally, a school choice problem with private and public schools contains the following ingredients:

- A finite set of **schools** $S = S^{pr} \cup S^{pu}$ where S^{pr} and S^{pu} denote the set of private and public schools, respectively. Note also that each school is either a private school or a public school, i.e., $S^{pr} \cap S^{pu} = \emptyset$.

- A finite set of **students** I.

- A **capacity** vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats at school $s \in S$.
- A student **preference profile** $P = (P_i)_{i \in I}$ where P_i is the strict preference relation of student $i \in I$ over the schools in S and the option of remaining unassigned. The latter option is denoted by $s_\emptyset$.
- A school **priority ordering** $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is the strict priority relation of school $s \in S$ over the students in I and the option of leaving a school seat unfilled. The latter option is denoted by $\emptyset$.

The option for a student to remain unassigned (i.e., $s_\emptyset$) is referred to as the **null-school** and it is assumed that $q_{s_\emptyset} = |I|$, i.e., that the model allows each student to remain unassigned. A school choice problem with private and public schools is denoted by $(S, I, q, P, \succ)$ and is, henceforth, referred to as a **grand problem**. For any given subset of schools $\bar{S} \subseteq S$ and any given subset of students $\bar{I} \subseteq I$, the related **subproblem** is denoted by $(\bar{S}, \bar{I}, q_{\bar{S}}, \bar{P}_{\bar{I}}|\bar{S}, \succ_{\bar{S}} |\bar{I})$. Here, $q_{\bar{S}} = (q_s)_{s \in \bar{S}}$ is the capacity vector of the schools in $\bar{S}$. Furthermore, $\bar{P}_{\bar{I}}|\bar{S}$ and $\succ_{\bar{S}} |\bar{I}$ are the preferences of the students in $\bar{I}$ over the schools in $\bar{S} \cup \{s_\emptyset\}$ and the priorities of the schools in $\bar{S}$ over students in $\bar{I}$, respectively. Note also that the preferences of the students in $\bar{I}$ over the schools in $\bar{S} \cup \{s_\emptyset\}$ can be different from the preferences of the students in $\bar{I}$ over the schools in $S \cup \{s_\emptyset\}$. To simplify notation, a subproblem $(\bar{S}, \bar{I}, q_{\bar{S}}, P_{\bar{I}}|\bar{S}, \succ_{\bar{S}} |\bar{I})$ is denoted by $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$ in the remaining part of the paper.

School $s \in S$ is **acceptable for student** $i \in I$ if school s is strictly preferred to the null-school, i.e., if $sP_is_\emptyset$. Student $i \in I$ is **acceptable for school** $s \in S \cup \{s_\emptyset\}$ if student i is strictly preferred to an unfilled seat, i.e., if $i \succ_s \emptyset$. By assumption, each student $i \in I$ is acceptable for the null-school, i.e., $i \succ_{s_\emptyset} \emptyset$ for each $i \in I$.

School priorities over subsets of students in I are assumed to be **responsive**. This means that for any school $s \in S$, any subset of students $J \subset I$ with $|J| < q_s$, and any two students $i, j \notin J$, the following two conditions hold:

(i) $(J \cup i) \succ_s J \iff i \succ_s \emptyset$,

(ii) $(J \cup i) \succ_s (J \cup j) \iff i \succ_s j$.

These conditions mean that as long as the capacity constraint of the school is not binding, the school prefers (i) an additional acceptable student to leaving a seat empty and (ii) a student with a higher priority to a student with a lower priority when filling an additional seat. For each student $i \in I$, let R_i denote the at least as good as relation associated with P_i, i.e.:

$sR_is' \iff sP_is'$ whenever $s \neq s'$.

Similarly, let for each school $s \in S$, $\succsim_s$ denote the at least as good as relation associated with $\succ_s$, i.e.:

$$J \succsim_s K \Longleftrightarrow J \succ_s K \text{ whenever } J \neq K.$$

For a given subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$, a **matching** is defined as a function $\mu : \bar{I} \to \bar{S} \cup \{s_\emptyset\}$ such that the number of students assigned to school $s \in \bar{S}$ does not exceed the capacity of school s, i.e., $|\mu^{-1}(s)| \leq q_s$ for all $s \in \bar{S}$ and $i \in \bar{I}$. With slight abuse of notation, in the rest of the paper, μ_i and μ_s are used instead of $\mu(i)$ and $\mu^{-1}(s)$, respectively. Let $\bar{\mathcal{M}} \equiv \mathcal{M}(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$ be the set of all possible matchings for subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$. All possible matchings under grand problem $(S, I, q, P, \succ)$ is denoted by $\mathcal{M}$, i.e., $\mathcal{M} \equiv \mathcal{M}(S, I, q, P, \succ)$.

2.2 Axioms

A number of axioms are used to compare and evaluate matchings. For simplicity, these axioms are defined for the grand problem $(S, I, q, P, \succ)$ but they are straightforwardly defined for any given subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$.

A matching μ is **non-wasteful** (for private schools) [for public schools] if there exists no student that prefers a school with an empty seat to her assigned school, i.e., if there exists no student $i \in I$ and school $s \in S$ ($s \in S^{pr}$) [$s \in S^{pu}$] such that $i \succ_s \emptyset$, $|\mu_s| < q_s$ and $sP_i\mu_i$.

A matching μ is **individually rational** if no student is assigned to a school that she finds unacceptable or she is unacceptable for. Formally, a matching μ is individually rational if $\mu_i R_i s_\emptyset$ and $i \succ_{\mu_i} \emptyset$ for all $i \in I$.

A matching μ is **fair** if whenever a student prefers some other student's assignment to her own, then the other student has a higher priority for that school than herself. Formally, μ is fair if for every $i, j \in I$, $\mu_j \in S$ and $\mu_j P_i \mu_i$ imply $j \succ_{\mu_j} i$.

A matching μ is **stable** if it is non-wasteful, individually rational and fair.

A matching μ is **mutually best** if there does not exist a student $i \in I$ and school $s \in S$ such that sP_is' for all $s' \in (S \cup \{s_\emptyset\}) \setminus \{s\}$, $i \succ_s i'$ for all $i' \in I \setminus \{i\}$ and $\mu_i \neq s$.

A student $i \in I$ prefers matching $\mu \in \mathcal{M}$ to matching $\nu \in \mathcal{M}$ if and only if she prefers μ_i to ν_i. A matching μ is Pareto dominated by matching ν if all students weakly prefer matching ν to matching μ and at least one student strictly prefers matching ν to matching μ, i.e., matching μ is **Pareto dominated** by matching matching ν if $\nu_i R_i \mu_i$ for all $i \in I$ and there exists at least one student $j \in I$ where $\nu_i P_i \mu_i$. A matching μ is **Pareto efficient** if there does not exist another matching ν which Pareto dominates μ.

A **mechanism** ϕ is a systematic procedure that selects a matching for any given subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$. The outcome selected by mechanism ϕ in subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$ is denoted by $\phi(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$, and the match of student $i \in \bar{I}$ and school $s \in \bar{S}$ are denoted by $\phi_i(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$ and $\phi_s(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$, respectively.

A mechanism ϕ is non-wasteful <individually rational> [fair]{stable} (Pareto efficient) <mutually best> if it selects a non-wasteful <individually rational> [fair] {stable} (Pareto efficient) <mutually best> matching for any given subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$.

3 The Sequential Student Assignment Game

The essential difference between the school choice model presented in the previous section and the standard model (Abdulkadiroğlu and Sönmez, 2003) is that the set of schools is partitioned into a set of private schools and a set of public schools, i.e., $S = S^{pr} \cup S^{pu}$ and $S^{pr} \cap S^{pu} = \emptyset$. This seemingly small deviation from the standard model makes a large difference when coupled with a sequential admission system. As will be exemplified in Section 5, many countries separate the admission to private schools from the admission to public schools. Students may, for instance, first have the option to apply to private schools and the private school admission is then followed by a centralized public school admission (e.g., the admission system used in Turkey between 2015–2017). Such two-stage admission processes also introduce many details that need to be analyzed. For example, if the admission to private schools proceeds the admission to public schools, can a student that is assigned to a private school also participate in the admission to public schools? If so, can the student "keep" her placement at the private school or does she have to "give up" the placement to participate in the public school admission?

To analyze the sequential school choice model under different sets of rules, a game in which play proceeds in a sequence of three rounds is considered. The analysis is restricted to the case where $\bar{S} \in \{S^{pr}, S^{pu}\}$, i.e., the case where the admission to private and public schools can be separated from each other in two different rounds of the game (as in, e.g., the above mentioned Turkish system). Because the considered admission systems, the game, and the theoretical findings can be explained without introducing a massive body of notation, a simpler exposition is adopted in the main body of the paper for ease of exposition. The interested reader is referred to Appendix A for the formal technical definitions.

The sequential student assignment game contains three separate rounds. In Round 1, only private or only public schools are available. The set of available schools in Round 1 is denoted by S^1 and, consequently, $S^1 \in \{S^{pr}, S^{pu}\}$. Students that participate in Round 1 are collected in the set I^1. Note that it may be the case that all students participate in Round 1, i.e., $I^1 = I$. The students in I^1 submit a rank order list over the schools in $S^1 \cup \{s_\emptyset\}$. If only private schools (public schools) are available in Round 1, then only public schools (private schools) are available in Round 2. The set of participating schools and students in Round 2 are denoted by S^2 and I^2, respectively. The students in I^2 submit a rank order list over the schools in $S^2 \cup \{s_\emptyset\}$. Because we only analyze systems in which the admission to private schools is separated from the admission to public schools, it follows that $S^2 = S \backslash S^1$ and, consequently, that $S^1 \cup S^2 = S$ and $S^1 \cap S^2 = \emptyset$. In Round 3, the students are assigned to schools. However, whether or not a student can participate in Round 2 and what alternatives the students have in Round 3 depend on the rules of the game.

The **rules of the game** are given by the mechanisms ϕ^1 and ϕ^2 that are used for assigning students to schools in Rounds 1 and 2, respectively, together with two correspondences, ψ and γ. Here, the correspondence ψ determines the set of students that are allowed to participate in Round 2, and the correspondence γ provides the alternatives for the students in Round 3. The correspondence ψ may or may not depend on the outcome of the mechanism ϕ^1. The analysis in this paper is restricted to two cases, namely when the correspondence ψ prescribes that:

- only students that are assigned to the null-school in Round 1 are allowed to participate in Round 2, i.e., $I^2 = \{i \in I : \phi_i^1(S^1, I^1, q^1, P^1, \succ^1) = s_\emptyset\}$,
- all students in I are allowed to participate in Round 2, i.e., $I^2 = I$.

Note that it is easy to extend the correspondence ψ to also include the intermediate cases in which some of the assigned students in Round 1 can participate in Round 2 and some of the unassigned students in Round 1 cannot participate in Round 2. All results presented in this paper still hold for these intermediate cases. The reason for restricting attention to the above situations is simply that they describe real-life admission systems (see Section 5).

The correspondence γ provides the alternatives for the students in Round 3. These alternatives do not only depend on the outcomes of the mechanisms ϕ^1 and ϕ^2, but also on if the student is allowed to participate in Round 2 or not. More precisely, the correspondence γ includes the following situations:

(1) If student i is not allowed to participate in Round 2, then she has to select the Round 1 assignment.

(2) If student i is allowed to participate in Round 2, then the selection can take three different forms:

   (a) student i can select the Round 1 or the Round 2 assignment if she participates in Round 2,

   (b) student i has to select the Round 2 assignment (the Round 1 assignment) if she (does not) participates in Round 2,

   (c) student i has to select the Round 1 assignment (the Round 2 assignment) if she participates in Round 2 and the Round 2 assignment is the null-school (distinct from the null-school).

Note that independently of if case (1) or (2) prevails, the correspondence γ is a subset of assignments from Rounds 1 and 2. Note also that even if the rules of the game are generally defined in the above, many existing sequential school admission systems fit perfectly in this framework. This will be discussed and exemplified in Section 5.

The sequential admission process described above and how it is related to the rules of the game can formally be described as:

Round 1. Each student $i \in I^1$ submits a rank order list P_i^1 over the schools in $S^1 \cup \{s_\emptyset\}$. The mechanism ϕ^1 assigns each student $i \in I^1$ to school $\phi_i^1(S^1, I^1, q^1, P^1, \succ^1)$ in $S^1 \cup \{s_\emptyset\}$.

Round 2. The correspondence ψ determines the set of students that are allowed to participate in Round 2. Each student that is allowed and willing to participate in Round 2, i.e., the students in the set $i \in I^2$, submit a rank order list P_i^2 over the schools in $S^2 \cup \{s_\emptyset\}$. The mechanism ϕ^2 assigns each participating student $i \in I^2$ to school $\phi_i^2(S^2, I^2, q^2, P^2, \succ^2)$ in $S^2 \cup \{s_\emptyset\}$.

Round 3. The correspondence γ provides the school alternatives for each student $i \in I$ based on the assignments in Rounds 1 and/or 2, and each student $i \in I$ selects one of the alternatives prescribed by γ.

One can think of students that are allowed to but do not wish to participate in Round $j \in \{1, 2\}$ as students that simply submit the null-school as their top choice when they report their preferences. In such case, students can always (and without loss of generality) be allowed to keep their Round 1 assignments. To simplify our analysis, a student who is allowed to but do not wish to participate in Round $j \in \{1, 2\}$ to rank $s_\emptyset$ as her top choice.

Throughout the paper, it is assumed that the moves in Round 1 are observed before Round 2 begins, and that the moves in Round 2 are observed before Round 3 begins. Participating students also move simultaneously within each round.

3.1 The Extensive Form Game

The above sequential process can also be regarded as an **extensive form game**. A detailed description of this game is provided in Appendix A. The informal definition follows next.

The game proceeds in three rounds. The **initial node** of the game, say h_1, can be thought of as Round 1, and it is based on subproblem $(S^1, I^1, q^1, P^1, \succ^1)$. In this node, a **subgame** is played. In this subgame, each student or, equivalently, each **player**, $i \in I^1$, reports a ranking P_i^1 over the schools in $S^1 \cup \{s_\emptyset\}$. The reported ranking belongs to the set of all possible (strict) rankings over the schools in $S^1 \cup \{s_\emptyset\}$ and it need not be truthful. The reported ranking P_i^1 also represents the **action** of student $i \in I^1$. The actions played by the students in I^1 at node h_1 together with the rules of the game will take the extensive form game to an **intermediate node** node, say h. Note that node h belongs to some set of possible nodes H^2 but exactly what node in H^2 that the game ends up in depends on the actions as well as the rules of the game. Node h is based on subproblem $(S^2, I^2, q^2, P^2, \succ^2)$ and it can be thought of as Round 2. A subgame, where each student $i \in I^2$ reports her ranking P_i^2 over the schools in $S^2 \cup \{s_\emptyset\}$, is then played at node h. Note here that the subgame played at node h is defined based on the play at the initial node h_1, i.e., the subgame at node h depends not only on the rules of the game but also on the **history** of the game. In a similar manner as in the above, the play at node h together with the rules of the game will take the extensive form game to a **penultimate node**, say $\bar{h}$. Node $\bar{h}$ can be regarded as Round 3 where each student selects a school (possibly the null-school) from the alternatives proposed to her. Again, the choices at node $\bar{h}$ depends on the history and the rules of the game. In this sense, the initial node h_1 and the strategies played in the various subgames define a unique penultimate node of the extensive form game.

Because the considered game is a sequential game, it is natural to adopt the notion of a **Subgame Perfect Nash Equilibrium** (SPNE, henceforth) as the equilibrium concept. Such an equilibrium is a strategy profile that induces a **Nash Equilibrium** in every subgame.

3.2 The Notion of Straightforwardness

It is well established in the school choice literature that students often can gain by misrepresenting their preferences over schools and that they sometimes use this possibility to their advantage (see, e.g., Abdulkadiroğlu et al., 2005; Dur et al., 2018; Pathak and Sönmez, 2018, 2013). Obviously, students can have different strategies also in the considered sequential game and not all of these strategies involve truthful reports over the available schools in each round. Because truth telling plays an important role in the school choice literature, we also need to introduce such a concept for the considered extensive form game. This is the notion of straightforwardness.

Definition 1. Given rules of the game $(\phi^1, \phi^2, \psi, \gamma)$, a student $i \in I$ plays a **straightforward** strategy if it involves the following actions:

Round 1: Student i reports her true ranking over the schools in $S^1 \cup \{s_\emptyset\}$.

Round 2: Student i reports her true ranking over the schools in $S^2 \cup \{s_\emptyset\}$ that are strictly more preferred to her Round 1 assignment (note that the Round 1 assignment might be $s_\emptyset$).

Round 3: Student i selects her most preferred school out of the schools prescribed by γ.

The notion of straightforwardness does not only capture the idea that students report truthful rankings over the schools, the notion also captures rationality in the sense that any student that play a straightforward strategy at the same time makes sure that their current situation in the game weakly improves according to her true preferences. For example, before Round 1 starts, no student is assigned to any school and, therefore, each student that plays a straightforward strategy reports a ranking over the schools that are (weakly) preferred to her current situation of being unassigned. Similarly, any student that participates in Round 2 and plays a straightforward strategy will weakly improve her current situation. For example, if the student is assigned the Round 2 assignment whenever it is distinct form the null-school, the student cannot lose anything by adopting the action to only report a ranking over the schools in $S^2 \cup \{s_\emptyset\}$ that are strictly preferred to her Round 1 assignment. The same holds for Round 3 where it, obviously, is rational to select the most preferred school of the available alternatives. Note also that the definition of a straightforward strategy covers all considered alternatives prescribed by the correspondence γ.

Finally, a Subgame Perfect Nash Equilibrium (SPNE) is straightforward if each student in I plays a strategy composed of only the straightforward actions.

4 General Equilibrium Analysis of Sequential School Admission Systems

Before investigating sequential systems used in different countries, a more general analysis of the sequential school admission game is provided. As will be demonstrated in this section, the rules of the game need to have very specific properties to guarantee the existence of a straightforward SPNE. Furthermore, there is a trade-off between existence of a straightforward SPNE and non-wastefulness. To reach these conclusions, we first establish two impossibility results related to the rules of the game. These negative results will pin down the exact conditions that are needed to define rules of the game that guarantee the existence of a straightforward SPNE. First, we focus on the impossibility results caused by the mechanisms ϕ^1 and ϕ^2.

Proposition 1. For any $i, j \in \{1, 2\}$ and $i \neq j$, if the mechanism ϕ^i is individually irrational <wasteful> [unfair], then for any (ψ, γ, ϕ^j) there exists a problem $(I, S, q, \succ, P)$ such that either none of the SPNE are straightforward or any straightforward SPNE induces an individually irrational <wasteful> [unfair] matching.

Proposition 1 implies that we can restrict attention towards mechanisms ϕ^1 and ϕ^2 that are individually rational, non-wasteful and fair. This is, in fact, a natural restriction because, in school choice contexts, assigning students to unacceptable schools, violating the priorities (obtained, e.g., from a centralized exam or predetermined criteria) or wasting some school seats are often highly criticized by the public.

In the following, we also consider mechanisms that are mutually best. Note that fairness and non-wastefulness imply mutual best, and that mutual best is weaker than fairness and non-wastefulness together. Furthermore, all well-studied school choice mechanisms, e.g., the deferred acceptance algorithm, the top trading cycles mechanism and the serial dictatorship mechanism, satisfy mutual best. The latter means that as long as one of the mentioned well-studied mechanisms are used, the result in Proposition 1 is strictly speaking not necessary for our conclusions.

Next, we focus on impossibility results caused by correspondences ψ or γ.

Proposition 2. Let $(\phi^1, \phi^2, \psi, \gamma)$ be the rules of the game. Then:

1. If ϕ^1 is non-wasteful and individually rational, ϕ^2 is non-wasteful, and (ψ, γ) prescribes that only students that are assigned the null-school in Round 1 can participate in Round 2, then there exists a problem $(I, S, q, \succ, P)$ such that there exists a unique SPNE that is not straightforward under the induced game.

2. If ϕ^1 is non-wasteful and individually rational, ϕ^2 is mutually best, and (ψ, γ) prescribes that all students are allowed to participate in Round 2 but have to select the Round 2 assignment if they participate in Round 2, there exists a problem $(I, S, q, \succ, P)$ such that there exists no straightforward SPNE under the induced game.

3. If ϕ^1 is non-wasteful and mutually best, ϕ^2 is non-wasteful, and (ψ, γ) prescribes that all students are allowed to participate in Round 2, then there exists a problem $(I, S, q, \succ, P)$ such that at least one of the SPNE outcomes is wasteful for the schools in S^1 under the induced game.

4. If ϕ^1 is non-wasteful and individually rational, ϕ^2 is mutually best, and (ψ, γ) prescribes that all students are allowed to participate in Round 2 and can select their most preferred outcomes from Rounds 1 and 2, then there exists a problem $(I, S, q, \succ, P)$ such that at least one of the SPNE outcomes is wasteful for the schools in S^2 under the induced game.

Given that ϕ^1 and ϕ^2 are individually rational, non-wasteful and mutually best, the first two parts of Proposition 2 imply that if not all agents are allowed to participate in Round 2 or are forced to give up their Round 1 assignment to participate in Round 2, then some problems do not have a straightforward SPNE. Moreover, the third part of Proposition 2 implies that if all agents are allowed to participate in Round 2, then some problems always lead to a wasteful SPNE outcome for the schools in S^1 (induced by a straightforward SPNE). Finally, the last part of Proposition 2 implies that if all students are allowed to participate in Round 2 without giving up their Round 1 assignment, then some problems always lead to a wasteful SPNE outcome for the schools in S^2 (induced by a straightforward SPNE).

The findings in Proposition 2 give important information about how the correspondences ψ and γ must be defined in order to guarantee the existence of a straightforward SPNE for any problem. More precisely, the first two parts of Proposition 2 imply that all students should be allowed to participate in Round 2 without giving up their Round 1 assignment. However, if such rules are adopted, for some problems, a straightforward SPNE induces an equilibrium outcome in which some school seats in Round 1 are wasted. Furthermore, if students are allowed to participate in Round 2 without giving up their Round 1 assignment, for some problems, a straightforward SPNE induces an equilibrium outcome in which some school seats in Round 2 are wasted. These findings suggest that there is some type of trade-off between the existence of a straightforward SPNE and non-wastefulness.

Consider now the correspondences ψ^* and γ^* where the former prescribes that all students are allowed to participate in Round 2, and the latter prescribes that any student that is assigned a school distinct from the null-school in Round 2 have to select the Round 2 assignment.[10] Let now the mechanisms ϕ^{1*} and ϕ^{2*} be any two mechanisms that are individually rational, non-wasteful and fair. Given the above insights, the rules of the game $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$ are not only

[10]In other words, only her Round 2 assignment will be available in Round 3.

well-defined but also the only remaining option that potentially can guarantee the existence of straightforward SPNE and at the same time reduce waste. As illustrated in the following theorem, this only remaining option will, in fact, do the job.

Theorem 1. Suppose that all schools in S have the same relative priorities over acceptable students and that the rules of the game are given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$. Then, for any sequential problem, the induced game has a straightforward SPNE and in any equilibrium outcome the seats for the schools in S^2 are not wasted.

Note now that when all schools have the same relative priorities over acceptable students, a constrained version of the serial dictatorship mechanism (or SD for short) is the unique individually rational, non-wasteful and fair mechanism. In particular, in this version of the serial dictatorship mechanism, the remaining student with the highest priority for all remaining schools will, in each step, select her highest ranked school among the available schools that find her acceptable. The following corollary now follows directly from Propositions 1–2 and Theorem 1.

Corollary 1. Suppose that all schools in S have the same relative priorities over acceptable students. Then, $(\text{SD}, \text{SD}, \psi^*, \gamma^*)$ are the unique rules of the game such that there exists a straightforward SPNE and any equilibrium outcome (induced by straightforward SPNE) is fair, individually rational and non-wasteful for schools in S^2.

Two remarks are in order. First, when all schools in S have the same relative ranking over acceptable students, under $(\text{SD}, \text{SD}, \psi^*, \gamma^*)$, playing straightforward strategy is a weakly dominant strategy for all students. This follows from the facts that a student cannot affect the assignments of students with higher priorities in any round and in Round 2 she only applies for the better schools than her Round 1 assignment.

Second, Theorem 1 holds whenever all schools in S have the same relative priorities over acceptable students. As demonstrated in the following example, the latter assumption is crucial for the result to remain true. More precisely, if schools are allowed to have heterogenous priorities, then the existence of a straightforward SPNE is not guaranteed even if the rules of the game are given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$.

Example 1. Let $S^1 = S^{pu} = \{s_1\}$, $S^2 = S^{pr} = \{s_2\}$, and $I = \{i_1, i_2\}$. Suppose further that each school has one available seat, and that student preferences are given by: $s_2 P_{i_1} s_1 P_{i_1} s_\emptyset$ and $s_1 P_{i_2} s_2 P_{i_2} s_\emptyset$. The school priorities are given by $i_1 \succ_{s_1} i_2 \succ_{s_1} \emptyset$ and $i_2 \succ_{s_2} i_1 \succ_{s_2} \emptyset$. Let the rules of the game be given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$. When both students play straightforward strategies, then student i_1 is assigned to school s_1, and student i_2 is assigned to school s_2. However, if i_1 deviates to a strategy in which she ranks the null-school $s_\emptyset$ above school s_1 in Round 1 and ranks s_2 above the null-school in any node in Round 2, and i_2 plays a straightforward strategy, then student i_1 is assigned to school s_2 and student i_2 is assigned to school s_1. Hence, there is no straightforward SPNE in this game. □

5 Sequential School Admission Systems

This section describes three different sequential school admission systems and shows how they should be interpreted in the considered sequential school admission framework.

5.1 The Old Turkish System

The Ministry of Education (MOE) in Turkey implemented some reforms in the Turkish high school admissions in 2014. In this system, students participated in two nationwide high school entrance exams in the eighth grade of elementary school. A Score for Placement (SFP) was calculated for each student by taking a weighted average of the test scores from the two nationwide entrance exams and the Grade Point Average (GPA). All public schools admissions are administered by MOE via a centralized clearinghouse, and priorities of public schools over students were based on the SFPs. Accordingly, all public schools had the same priorities over the students. However, private schools was not included in the centralized match as MOE lacks jurisdiction over private school admissions. Note, however, that private schools often had different sets of acceptable students in the sense that they had different views on the lowest acceptable SFPs (the so-called cut-off scores).[11]

The old Turkish system worked as follows. In Round 1, each student submitted a ranking over 15 public schools. Students were then assigned to public schools based on the serial dictatorship mechanism where the priorities were given by the SFPs. In Round 2, all students were allowed to participate independently of the outcome in Round 1. Participating students applied to private schools and were admitted to private schools in a decentralized way. In Round 3, the students selected their most preferred outcome from Rounds 1 and 2. Formally, this means that the correspondence ψ prescribes that $I^2 = I$ and that the correspondence γ is described by case (2a) from Section 3. Appendix C demonstrates that the decentralized admission game for private schools has a unique SPNE outcome which is equivalent to the outcome of the (constrained) serial dictatorship mechanism under true preferences and true cut-off scores.[12]

It is next demonstrated that any SPNE outcome of the old Turkish system is fair and individually rational but that it need not be non-wasteful. The latter conclusion is demonstrated by means of an example and it intuitively holds because students need not give up their Round 1 assignments to participate in Round 2. Consequently, if some student that was assigned a school in Round 1 selects her Round 2 assignment, a school seat from Round 1 may be wasted.

Proposition 3. Any SPNE outcome of the old Turkish system is individually rational and fair.

[11] Most private schools also form their priorities via SFPs. There are only few private schools (mostly international schools) that use different weights.

[12] In particular, we consider a sequential game in which private schools announce their cut-off scores and students that apply to schools can hold at most one offer at a time.

Example 2. To demonstrate that the old Turkish system is wasteful, let $S^1 = S^{pu} = \{s_1\}$, $S^2 = S^{pr} = \{s_2\}$, and $I = \{i_1, i_2, i_3\}$. Suppose further that each school has two available seats, and that student preferences are given by: $s_2 P_{i_1} s_1 P_{i_1} s_\emptyset$, $s_1 P_{i_2} s_2 P_{i_2} s_\emptyset$ and $s_1 P_{i_3} s_2 P_{i_3} s_\emptyset$. The school priorities are given by $i_1 \succ_s i_2 \succ_s i_3 \succ_s \emptyset$ for all $s \in S$. In this example, a strategy where each student submits her true preferences over the available schools in any node in Rounds 1 and 2, and then selects her most preferred school in any node in Round 3 is a SPNE. Given such strategy, students i_1 and i_2 are admitted to school s_1 in Round 1, and students i_1 and i_2 are admitted to school s_2 in Round 2. Consequently, the outcome in Round 3 is that student i_1 is assigned to school s_2, student i_2 is assigned to school s_1, and student i_3 is assigned to the null-school $s_\emptyset$. Hence, one seat at school s_1 and and one seat at school s_2 is wasted. □

Finally, we note that Proposition 3 and Example 2 hold also when private school assignment is done before the public school assignment as long as the correspondences ψ and γ are defined as in the old Turkish system. Hence, the wastefulness in the assignment cannot be solved by just changing the order of assignment.

Because of the option of holding offers in both rounds and unrestricted entrance to Round 2, one can see similarities between the old Turkish system and parallel assignment systems in which students can apply to public and private schools at the same time and then pick the best offer she receives. In this sense, the old Turkish system and the parallel assignment system adopted by most Swedish municipalities have common equilibrium properties.

5.2 The New Turkish System

As demonstrated in the previous example, the old system resulted in a large amount of vacant seats in public schools. This led to many rounds of reassignments which hardly resolved the problem. To mitigate these issues, MOE replaced the admissions system in 2015. The main difference compared to the old system was that the roles of the private and public schools were reversed, and that the rules related to which students that are allowed to participate in Round 2 (i.e., the correspondence ψ) was changed. More precisely, in Round 1, students applied to private schools and the admission was, again, organized in a decentralized fashion. Only students that remained unassigned after Round 1 were allowed to participate in Round 2. The participating students submitted a ranking over 25 public schools. Formally, this means that the correspondence ψ prescribes that $I^2 = \{i \in I : \phi_i^1(S^1, I^1, q^1, P^1, \succ^1) = s_\emptyset\}$ and that correspondence γ is described by case (1) from Section 3. Note also that this procedure will assign each student to at most one school in Round 3. As demonstrated next, in contrast to the old Turkish system, any SPNE in the new Turkish system is stable.

Proposition 4. Any SPNE outcome of the new Turkish system is stable, i.e., non-wasteful, individually rational and fair.[13]

[13] Since all schools rank the acceptable students in the same order, there is a unique stable matching.

As in the old Turkish system, the decentralized admission game for private schools has a unique SPNE outcome which is equivalent to the outcome of the (constrained) serial dictatorship mechanism under true preferences and true cut-off scores (see Appendix C) and the result holds also when reversing the roles of the public and private schools in the assignment process.

Although any SPNE outcome of the new Turkish system is stable, Proposition 2 implies that for some problems under the new Turkish system there does not exist a straightforward SPNE. To be more specific, in the new system, students are faced with a risky decision: either register for the best available private school and do not apply in the second round, or give up your private school assignment and try your chances for a better public school (this concern was also ventilated in a letter from a parent to the President of Republic of Turkey, see footnote 6). Another interesting implication of Propositions 2 and 4 is the following corollary.

Corollary 2. Suppose that all schools in S have the same relative priorities over acceptable students. Then, the new Turkish system's rules are the unique rules of the game such that any SPNE outcome is stable.

Finally, when we compare the equilibrium outcomes under the old and new Turkish systems, Propositions 3 and 4 imply the following corollary.

Corollary 3. Suppose that all schools in S have the same relative priorities over acceptable students. Then, any equilibrium outcome of the old Turkish system is (weakly) Pareto dominated by the unique equilibrium outcome of the new Turkish system.

5.3 The Botkyrka System

In Sweden, the municipalities determine the admission process for public schools even if they are guided by national legislation (*Skollagen*). Admission to public schools is centralized within each municipality and the priorities for public schools are decided based on relative distance to schools. The latter means that public schools typically have different priorities over students. Admission to private schools is decentralized but is also guided by legislation. The municipality of Botkyrka (south-west of Stockholm) will be used to illustrate the Swedish system.[14]

In the Botkyrka system, parents rank schools in a centralized online application system using a drop down menu. This menu contains all private and public schools in the municipality. The system allows parents to rank at most three schools in total. However, an application to a private school will only be considered if it is top-ranked (i.e., the most preferred school according the the students submitted ranking).[15] In Round 1, the application for each student that ranks a private school as a top-choice is forwarded to that private school. All private schools have a waiting list, and priority is given on first-come-first-served basis. The private schools then match the

[14] See Andersson (2017) for a detailed description of the Swedish school choice system, and Kessel and Olme (2018) for a more detailed description of the Botkyrka system.

[15] See also the discussion in Section 6.

applications to their waiting lists, and returns the names of the students they wish to admit to the centralized application system. In Round 2, only students that remain unassigned after Round 1 are allowed to participate. The ranking of public schools for these students are forwarded to a centralized clearinghouse and the admission is done via the (student proposing) deferred acceptance algorithm. This procedure will assign each student to at most one school in Round 3.

Note that the correspondences ψ and γ are defined as in the new Turkish system but that the mechanisms ϕ^1 and ϕ^2 as well as the priority orders differ from the new Turkish system. This will also have the consequence that the properties of any SPNE outcome differs between the new Turkish system and the Botkyrka system. More precisely, a SPNE need not be non-wasteful for public schools in the Botkyrka system unless there is a SPNE where students play straightforward actions in Round 2.[16] Furthermore, a SPNE outcome of Botkyrka system need not be fair. The latter result is illustrated by means of an example.

Proposition 5. Any SPNE outcome of Botkyrka system is non-wasteful and fair for private schools, and individually rational. Moreover, when students play true preferences in Round 2, any SPNE outcome is non-wasteful for the public schools.

Example 3. Let $S^1 = S^{pr} = \{s_1\}$, $S^2 = S^{pu} = \{s_2, s_3\}$, and $I = \{i_1, i_2, i_3\}$. Suppose further that each school has one available seat, and that student preferences are given by: $s_2 P_{i_1} s_1 P_{i_1} s_\emptyset$, $s_2 P_{i_2} s_3 P_{i_2} s_\emptyset$, and $s_3 P_{i_3} s_2 P_{i_3} s_\emptyset$. The school priorities are given by $i_1 \succ_{s_1} i_2 \succ_{s_1} i_3 \succ_{s_1} \emptyset$, $i_3 \succ_{s_2} i_1 \succ_{s_2} i_2 \succ_{s_2} \emptyset$, and $i_2 \succ_{s_3} i_3 \succ_{s_3} i_1 \succ_{s_3} \emptyset$. In this example, straightforward play by all students is a SPNE. The outcome of this strategy is that student i_1 is assigned to school s_1, student i_2 is assigned to school s_2, and student i_3 is assigned to school s_3. But this outcome is not fair since student i_1's priority is not respected at public school s_2 since s_2 is assigned to student i_2 but $s_2 P_{i_1} s_1$ and $i_1 \succ_{s_2} i_2$. □

Proposition 5 stated that any SPNE outcome is non-wasteful for the public schools when students submit their true preferences in Round 2 (this is a weakly dominant strategy in each subgame in that round). In the following example, it is illustrated that one may end up with wasteful equilibrium outcomes for the public schools when students play other equilibrium strategies.

Example 4. Let $S^1 = S^{pr} = \{s_1\}$, $S^2 = S^{pu} = \{s_2, s_3\}$, and $I = \{i_1, i_2\}$. Suppose further that each school has one available seat, and that student preferences are given by: $s_2 P_{i_1} s_1 P_{i_1} s_3 P_{i_1} s_\emptyset$ and $s_3 P_{i_2} s_2 P_{i_2} s_\emptyset$. The school priorities are given by $i_1 \succ_{s_1} i_2 \succ_{s_1} \emptyset$, $i_2 \succ_{s_2} i_1 \succ_{s_2} \emptyset$, and $i_2 \succ_{s_3} i_1 \succ_{s_3} \emptyset$. In this example, the following strategy profile is a SPNE: (a) both students play their straightforward action in Round 1, (b) both students rank the school in which they have the highest priority in Round 2 at the top whenever the other student also participates in Round 2, and rank their most preferred school in Round 2 at the top otherwise. In the outcome related to

[16] If we focus on the simultaneous preference revelation game in which students submit a single preference list, any Nash equilibrium outcome is non-wasteful, individually rational and fair for private schools.

this SPNE, student i_1 is assigned to school s_1 and student i_2 is assigned to school s_3. Hence, the seat at s_2 is wasted. □

Although, we may end up with wasteful and/or unfair equilibrium outcomes under Botkyrka system, both Botkyrka and the new Turkish systems are structurally the same. In particular, if we consider Botkyrka system under homogeneous priorities for all schools, then any equilibrium outcome will be stable (see the proof of Proposition 4).

6 Empirical Evaluation

From the theoretical findings in the previous sections, we know that the new Turkish system is the only of the investigated systems that is non-wasteful for all schools in any SPNE even if some of the other systems are non-wasteful for private *or* public schools. We also know that the only rule that induces all straightforward SPNE with non-wasteful outcomes for schools in Round 2 is defined in Section 4 and given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$, but that this rule produces a wasteful outcome. Because we are interested in straightforward SPNE, the aim of this section is to quantify the wastefulness for various rules of the game and, furthermore, to produce a measure of the exact trade-off between the existence of a straightforward SPNE and wastefulness. To achieve this, the 2015 data for the school admission of 6-year-old children in Botkyrka municipality will be analyzed.[17]

The reason for using the Botkyrka admission data to evaluate the theoretical findings is that the data is ideal for our purposes since all applications are handled via a centralized application system even if admissions to private schools are decentralized (see Section 5.3). This also means that the data contains information about how students rank public schools in relation to private schools. This type of information is impossible to get if students submit their rankings to private and public schools in separate application systems. We are not aware of any other data set on school choice with this specific property. However, for such information to be valuable, there must be a "sufficiently large" variation in the submitted rankings. For example, if students that rank private schools rank exactly one private school and, in addition, always as their top-choices, then there is almost no variation in the rankings since only two types of rankings then are contained in the data set, i.e.:

(a) all three ranked schools are public schools,

(b) the top-ranked school is a private school and the other two ranked schools are public schools.

[17]The current practice in Botkyrka is not immune to manipulation by students. However, students do not have any incentive to misreport the relative ranking of public schools listed. In particular, it is a (weakly) dominated strategy to change the true ranking of public schools. Given that our main goal is to quantify wastage under various models rather than performing a full-fledged welfare analysis, it suffices to restrict attention to schools that are reported as acceptable. Clearly, declaring a truly unacceptable school as acceptable is also dominated.

Recall now from Section 5.3 that only rankings of type (a) and (b) play a role in the Botkyrka admission system. However, this information is *not* transmitted to the parents and this fact gives a variation in the types of reported rankings that can be used to evaluate different admission systems.[18] In fact, in the 2015 admission data, it can be seen that only 85.5% of all students submit a ranking of type (a) or (b). For example, 7.1% of all students rank a private school as their third choice but are unaware of the fact that they never can be assigned to that school (since it not is top-ranked).

In 2015, there were 24 schools in Botkyrka municipality including 18 public and 6 private schools. These schools had between 10 and 76 school seats each. A total number of 1,109 students applied to the schools but the analysis only considers 1,033 of these students. Reasons for excluding students are, e.g., that the student no longer lived in Botkyrka at the end of the admission period, that the student decided to start at a school outside of Botkyrka, etc. The data contains all information needed to evaluate the theoretical results except a unified priority order based on, e.g., test scores. Because no such priority order exists in Botkyrka, the analysis is based on the average of 1,000 randomly drawn priority orders.[19]

We consider two versions of three different models of sequential admission systems. The two versions are defined based on if admission to private schools is conducted before the admission to public schools (the **private–public** version, henceforth) or vice versa (the **public–private** version, henceforth). Independently of which version of the model that is analyzed, the mechanisms ϕ^1 and ϕ^2 will be defined by the serial dictatorship mechanism. In this case, the new Turkish system and the Botkyrka system coincide. The following models will be considered in its two different versions:

Model 1. The rules of the game are as in the new Turkish system.[20]

Model 2. The rules of the game are as in the old Turkish system.

Model 3. The rules of the game are given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$ as defined in Section 4.

The following example illustrates that the above three models can generate distinct matchings given that reported rankings can differ from (a) and (b) in the above.

Example 5. Let $I = \{i_1, i_2, i_3, i_4, i_5\}$, $S^{pr} = \{s_1, s_2, s_3\}$ and $S^{pu} = \{s_4, s_5\}$. All schools have exactly one school seat. Suppose further that the common priority order for all schools $s \in S$ is given by $i_1 \succ_s i_2 \succ_s i_3 \succ_s i_4 \succ_s i_5 \succ_s \emptyset$. The rankings of the students are given by

[18] It is not very difficult to see that if all parents would report rankings only according to (a) and (b), then all rules considered in this section will generate identical matchings. As later will be obvious from Example 5 and the empirical analysis, this is not the case given that some parents report rankings different from (a) and (b).

[19] More precisely, the priority orders are given by a uniform random permutation of the vector containing the 1,033 students in the data set.

[20] Recall from the above that the rules of the game are identical for the new Turkish system and the Botkyrka system when the mechanisms ϕ^1 and ϕ^2 are given by the serial dictatorship mechanism.

$s_1 P_{i_1} s_2 P_{i_1} s_5 P_{i_1} s_\emptyset$, $s_1 P_{i_2} s_4 P_{i_2} s_2 P_{i_2} s_\emptyset$, $s_4 P_{i_3} s_3 P_{i_3} s_5 P_{i_3} s_\emptyset$, $s_3 P_{i_4} s_5 P_{i_4} s_4 P_{i_4} s_\emptyset$, and $s_1 P_{i_5} s_2 P_{i_5} s_3 P_{i_5} s_\emptyset$. Students are assumed to play straightforward strategies. In particular, under Model 2, students rank all schools in Round 2 truthfully. Given this, the private–public version of Models 1–3 generate the following matchings where μ_j^m denotes the matching of student i_j at Model m:

$$\begin{aligned}
\mu^1 &= (\mu_1^1, \mu_2^1, \mu_3^1, \mu_4^1, \mu_5^1) = (s_1, s_2, s_3, s_5, s_\emptyset),\\
\mu^2 &= (\mu_1^2, \mu_2^2, \mu_3^2, \mu_4^2, \mu_5^2) = (s_1, s_4, s_3, s_\emptyset, s_\emptyset),\\
\mu^3 &= (\mu_1^3, \mu_2^3, \mu_3^3, \mu_4^3, \mu_5^3) = (s_1, s_4, s_3, s_5, s_\emptyset).
\end{aligned}$$

Consider first Model 1. Because all students play straightforward strategies and all students have ranked at least one private school, all students participate in Round 1 and report their true rankings over the private schools. By the priority order, schools s_1, s_2 and s_3 are in Round 1 assigned to students i_1, i_2, and i_3, respectively. These students are not allowed to participate in Round 2. Note next that student i_5 does not rank any public school and will, consequently, not participate in Round 2. Hence, only student i_4 is allowed and willing to participate in Round 2, and student i_4 is, consequently, assigned her most preferred public school, i.e., school s_5. This gives matching μ^1.

Consider next Model 2. Here, students assigned in Round 1 can keep their assignment and participate in Round 2 and decide which placement to select in Round 3. Again, because all students play straightforward strategies and all students have ranked at least one private school, all students participate in Round 1 and report their true rankings over the private schools. By the priority order, schools s_1, s_2 and s_3 are in Round 1 assigned to students i_1, i_2, and i_3, respectively. In Round 2, all students except student i_5 will participate by straightforwardness (in fact, student i_5 ranks $s_\emptyset$ as her top choice). By the priority order, schools s_4 and s_5 are assigned to students i_2 and i_1, respectively. The students then select the best schools of the schools they have been assigned in the two rounds. For example, student i_2 is assigned school s_2 in Round 1 and school s_4 in Round 2. Because $s_4 P_{i_2} s_2$, student i_2 selects school s_4. This generates matching μ^2.

Consider now Model 3. Because students that participate in Round 1 can decide after seeing their Round 1 assignment weather to participate in Round 2 or not, all students that have applied to a private school will participate in Round 1 by straightforwardness. By the same arguments as in the above, schools s_1, s_2 and s_3 are assigned to students i_1, i_2 and i_3, respectively. Because students play straightforward strategies, students i_2 and i_3 will participate in Round 2 since $s_4 P_{i_2} s_2$ and $s_4 P_{i_3} s_3$, respectively. Hence, all students except students i_1 and i_5 participate in Round 2 (in fact, students i_1 and i_5 rank $s_\emptyset$ as their top choice). Note, however, that students i_2 and i_3 will, in Round 2, by straightforwardness, only report the public schools that they find strictly better than their match from Round 1. This means that students i_2 and i_3 only report school s_4 in Round 2. By the priority order, school s_4 is first assigned to student i_2 in Round 2. Then student i_3 is assigned school $s_\emptyset$ (consequently, school s_3 will be the final matching for student i_3). Finally, student i_4 is assigned school s_5. This gives matching μ^3. □

Because the considered models are based on different rules, they also imply different round participation for the students (this is also apparent from Example 5). However, independent of which model that is analyzed, all students that have applied to a private school (public school) will participate in Round 1 for the private–public setting (public–private setting) when they play straightforward strategies. This follows since these students cannot lose anything by such play.[21] Table 1 states the round participation rates for the different models.

Table 1: *Round participation for the different models (averages over 1,000 simulations). Standard deviation within brackets.*

Version	Model	Round 1 participation	Round 2 participation
Private–Public	Model 1	19.65% (0.00)	90.19% (0.13)
	Model 2	19.65% (0.00)	99.71% (0.00)
	Model 3	19.65% (0.00)	95.58% (0.34)
Public–Private	Model 1	99.71% (0.00)	4.120% (0.37)
	Model 2	99.71% (0.00)	19.65% (0.00)
	Model 3	99.71% (0.00)	16.67% (0.41)

By the above conclusion that each student that has applied to a school in S^1 always participates in Round 1, it follows from Table 1 that 19.65% of all students have applied to (at least) one private school and 99.71% of all students have applied to (at least) one public school. Note also that the Round 2 participation is lower for Model 1 than the other two models. This follows since students that are assigned a school distinct from the null-school in Round 1 are not allowed to participate in Round 2. Similarly, in Model 3, all students are allowed to participate in Round 2 but a student will only do so if she ranks a school that participates in Round 2 (e.g., a public school in the private–public setting) higher than her Round 1 assignment. This increases the Round 2 participation compared to Model 1, but the participation rate is still strictly lower in Model 3 compared to Model 2 since students in Model 2 never can lose anything by participating in Round 2. Furthermore, since all students that have applied to a private school (public school) participate in Round 1 for the private–public setting (public–private setting), the standard deviation for the Round 1 participation is always zero.

Table 2 describes the percentage of students assigned to their first ranked, second ranked, and third ranked school. In addition, the table also specifies the percentage of students not assigned to any of their three ranked schools (i.e., unassigned students). As can be seen from the table, Model 3 assigns the largest proportion of students to their most preferred schools. This finding is a direct consequence of the rule that all students are allowed to participate in Round 2 and the strategy choice to do so only if they could potentially be assigned a more preferred school. Such a rule and such strategy choices do not prevail in Models 1 and 2. From Table 2 it is also clear that

[21]This is true for homogeneous priorities.

Models 1 and 2 has the lowest and the highest rate of unassigned students, respectively.[22] As will be obvious later, this finding follows from the fact that Model 1 (Model 2) is the least wasteful (the most wasteful) model and, therefore, students will on average hold fewer (more) school seats in Round 3 in comparison to the other models and this, necessarily, reduces (increases) the number of unassigned students.

Table 2: *Placements for the different models (averages over 1,000 simulations). Standard deviation within brackets.*

Sequence	Model	Rank 1	Rank 2	Rank 3	Unassigned	Sum
Private–Public	Model 1	83.62% (0.75)	8.87% (0.66)	4.84% (0.53)	2.67% (0.30)	100.00%
	Model 2	81.75% (0.61)	6.21% (0.60)	2.78% (0.46)	9.26% (0.29)	100.00%
	Model 3	85.24% (0.68)	6.63% (0.57)	2.56% (0.42)	5.57% (0.39)	100.00%
Public–Private	Model 1	79.87% (0.82)	10.2% (0.65)	4.41% (0.52)	5.57% (0.42)	100.00%
	Model 2	81.75% (0.61)	6.21% (0.60)	2.78% (0.46)	9.26% (0.29)	100.00%
	Model 3	85.29% (0.63)	4.87% (0.55)	2.30% (0.42)	7.54% (0.40)	100.00%

To quantify the **degree of wastefulness** at a given matching μ, the following measure will be adopted for each school s that has not reached their capacity at the given matching μ, i.e., for each school s with $|\mu_s| < q_s$ at matching μ:

$$\text{WASTE}_s = \text{number of students } i \text{ where } i \succ_s \emptyset \text{ and } sP_i\mu_i.$$

In, e.g., Model 2 in Example 5, the waste measure for school s_2 is $\text{WASTE}_{s_2} = 1$ since student i_5 is unassigned, no student is assigned to school s_2, and both school s_2 and student i_5 would prefer that they are matched. Note also that this measure implies that a student can be included in the measure WASTE_s for multiple schools $s \in S$, e.g., if the student is unassigned but would like to be matched to several distinct schools with unfilled school seats. In this sense, the measure of wastefulness, for a given school, is the number of students that would prefer to be matched to that school rather than their current match given that the school has empty school seats and prefer the student over an empty seat $\emptyset$. The aggregated waste measures for private and public schools are given by:

$$\begin{aligned} \text{WASTE}_{pr} &= \sum_{s \in S^{pr}} \text{WASTE}_s, \\ \text{WASTE}_{pu} &= \sum_{s \in S^{pu}} \text{WASTE}_s, \end{aligned}$$

respectively. The total wastefulness for a given model is defined as the sum of wastefulness for

[22] In particular, the straightforward strategy under Model 3 is one of the straightforward strategies under Model 2. However, in order to emphasize the differences between models we focus on the straightforward strategy under which students rank all schools truthfully in Round 2.

public and private schools, i.e.:

$$\text{Total WASTE} = \text{WASTE}_{pu} + \text{WASTE}_{pr}.$$

The waste measures for the different models are displayed in Table 3.

Table 3: *The wastefulness measure for the different models (averages over 1,000 simulations). Standard deviation within brackets.*

Sequence	Model	WASTE_{pr}	WASTE_{pu}	Total WASTE
Private–Public	Model 1	0.000 (0.00)	31.70 (8.18)	31.70
	Model 2	87.38 (5.34)	254.6 (18.8)	342.0
	Model 3	75.68 (5.84)	0.000 (0.00)	75.68
Public–Private	Model 1	84.31 (4.53)	0.000 (0.00)	84.31
	Model 2	254.6 (18.8)	87.38 (5.34)	342.0
	Model 3	0.000 (0.00)	256.7 (15.5)	256.7

As expected, Model 1 is the least wasteful model in total. This conclusion holds for both the private–public and the public–private setting, and is in line with the equilibrium predictions in Sections 4 and 5. For example, Proposition 4 predicts that Model 1 is non-wasteful in any SPNE. However, straightforward play, as assumed in this section, is not a SPNE in Model 1 and it is, therefore, expected that the waste measure for Model 1 in Table 3 is greater than zero. In Model 3, the waste is zero for the public schools in the private–public setting and zero for the private schools in the public–private setting. This is exactly according to the equilibrium prediction in Theorem 1 (recall that students play straightforward strategies in this section). Because Model 2 is the only model where students are allowed to participate in Round 2 without any restrictions, it is expected that Model 2 performs the worst of the three models in both the private–public and the public–private setting.

Note also that the waste is, on average, almost three times higher in the public–private setting than in the private–public setting for any given model. This is a direct consequence of the fact that students apply to more public schools (recall from the above that only 19.65% of the students apply to a private school whereas 99.71% of the students apply to a public school). Maybe the most important conclusion is that a social planner has to pay a quite high price in terms of wastefulness in order to implement an equilibrium where students play straightforward strategies. This can be seen by comparing the outcomes in terms of waste for Model 1 and 3 where the waste is almost three times higher in the latter model (recall, again, that the waste is zero for Model 1 in equilibrium by Proposition 4). In this sense, there is a trade-off between the existence of a straightforward SPNE and waste. However, the rule used in Model 3 does not lead to wasted school seats for the schools in S^2 by Theorem 1, and the above findings then suggest that by defining S^1 as the least popular set of S^{pu} and S^{pr} in terms of applications (i.e., private schools in the Botkyrka case), the total waste can be minimized. Of course, the decision related to the which of the two sets that are "least popular" cannot depend on the reports of the students in order to avoid introducing further manipulation possibilities. Instead, it must be based on some exogenous factor or historical data. In the Botkyrka case, for example, in the last few years, around 20% of the students have applied to private schools but close to 100% have applied to public schools. In such a case, it is not unrealistic to believe that private schools will be less popular in terms of applications also in the next application period.

7 Conclusions

This paper has studied sequential school choice systems formulated as general multi-stage sequential games. One of the main takeaway messages of the paper is that there is a trade-off between the existence of a straightforward SPNE and non-wastefulness. More precisely, earlier studies has established that there are trade-offs that involve existence of equilibria and "efficiency notions" both in systems with parallel school assignments and in systems with sequential admissions (Doğan and Yenmez, 2018a; Dur and Kesten, 2018; Ekmekci and Yenmez, 2014; Manjunath and Turhan, 2016, see, e.g.,). What separates this study from these papers is that the considered general sequential setting enables us to demonstrate that there is a unique set of rules for two-stage admission in school choice that guarantee the existence of a straightforward SPNE which, at the same time, reduces the waste of school seats.

Given the theoretical and empirical observations that manipulation is a problem in school choice, (see, e.g. Abdulkadiroğlu et al., 2005; Abdulkadiroğlu and Sönmez, 2013; Agarwal and Somain, 2014; Dur et al., 2018; Pathak and Sönmez, 2018, 2013), the results in this paper also imply that there exists a rule that induces truthfulness (defined by the notion of straightforwardness) also for sequential settings. As already concluded in the above, using this rule comes at a cost in terms of wastefulness. However, as revealed by the empirical analysis, this waste can be reduced by designing the school application sequence in a fashion such that students apply to either public or private schools first, depending on which set of schools that have the fewest number of (expected) applicants. The policy implication that comes out of this analysis is that if a social planner believes that truthful reporting is important, then the planner has to accept that school seats will be wasted but the waste can be reduced by a clever design. Moreover, this unique design gives incentives for the schools assigned in Round 1 (private schools) to join the assignment in Round 2 (centralized public school admission).[23] Eventually, it will result in a unified admission system without coercing any school.

Because this paper is one of the first to study sequential school choice, there is room for future research. For example, the emphasis in this paper has been to introduce a notion of truthfulness in sequential school choice and to analyze the conditions needed for the existence of a

[23] We prove this result in Proposition 6 Appendix B.

straightforward SPNE in a very general framework and the implications on, e.g., wastefulness. Of course, there is a number of other axioms, notions and potential trade-offs that may be policy relevant, and it is therefore difficult to draw too general conclusions based on this study alone. Hence, the findings and the analysis in this paper should be seen as a first step and not as a complete solution as more research is needed to fully understand complex sequential school choice systems.

Appendices

A The Extensive Form Game

This appendix gives a formal description of the extensive form game that previously was informally introduced in Section 3. The appendix first defines the extensive form game in general terms and then describes how this game fits in the considered sequential school admission framework.

An **extensive form matching mechanism** is a list $E = (I, H, M, \pi)$ where I is the set of **players**, H is the set of **histories** (nodes), M is the **strategy space**, and π is the **outcome function**.[24] The strategy space is composed of the **actions** that can be played at every history: $M = \Pi_{i\in I} M_i$ and $M_i = \Pi_{h\in H} M_i^h$ for every $i \in I$ where M_i^h is the actions that can be played by player $i \in I$ at history $h \in H$.[25] For a given **strategy profile** $m \in M$, we denote the strategy of player i with m_i. Hence, $m = (m_i)_{i\in I}$. Let h_1 be the **initial node** and H^T be the set of **penultimate nodes**. The outcome function $\pi : H^T \to \mathcal{M}$ gives a matching for each penultimate node. Note that, given the initial node h_1, every strategy profile $m \in M$ defines a unique penultimate node. Let h_m be the penultimate node defined by strategy m. With a slight abuse of notation, for each strategy profile m, we use $\pi(m)$ instead of $\pi(h_m)$. A **preference profile** P and extensive form matching mechanism E constitute a **game** $\Gamma = (I, H, M, \pi, P)$.

Each node $h \in H \setminus H^T$ identifies a **subgame** $\Gamma(h) = (I, H(h), M(h), \pi_h, P)$ where h is an initial node, $H(h) = \{h' \in H | h' \text{ follows } h\}$, $M(h) = \Pi_{i\in I} \Pi_{h'\in H(h)} M_i^{h'}$, and $\pi_h(m) = \pi(\bar{h})$ where strategy $m \in M(h)$ specifies penultimate node $\bar{h} \in H^T$ starting from node h. Let $m(h) \in M(h)$ be the strategy of subgame $\Gamma(h)$ related with $m \in M$. A **Subgame Perfect Nash Equilibrium** (SPNE) is a strategy that induces a **Nash Equilibrium** in every (proper) subgame. That is, the strategy profile $m^* \in M$ is a subgame perfect Nash equilibrium if for all $h \in H$ and each player $i \in I$, it holds that $\pi_h(m^*(h)) R_i \pi_h(m'_i, m^*_{-i}(h))$ for every $m'_i \in \Pi_{h'\in H(h)} M_i^{h'}$.

We next describe how the above defined game can be translated into the sequential school

[24] We use a similar notation as Romero-Medina and Triossi (2014).

[25] Note that the analysis allows for the possibility that $M_i^h = s_\emptyset$ for some $i \in I$ and for some $h \in H$. In particular, $M_i^h = s_\emptyset$ means that player $i \in I$ is not active at node h.

admission framework considered in this paper. For this purpose, denote the histories in Round k of the sequential admission process by H^k. Because the admission process starts in Round 1, the initial node h_1 is unique and belongs to the set $H^1 = \{h_1\}$. Moreover, because the sequential admission process consists of three rounds, it follows that the set of penultimate nodes is given by $H^T = H^4$. Let the set of "active students" (discussed below) in node h be given by I^h. With slight abuse of notation, if $i \notin I^h$ for some $h \in H$, we suppose that $M_i^h = s_\emptyset$ (see also footnote 25). The set of active schools in Rounds 1 and 2 are given by S^1 and S^2, respectively. The sets of all possible (strict) rankings over $S^1 \cup \{s_\emptyset\}$ and $S^2 \cup \{s_\emptyset\}$ are given by $\mathcal{P}^1$ and $\mathcal{P}^2$, respectively. Consequently, the actions that can be played by any student $i \in I^{h_1}$ at node h_1 are given by $M_i^{h_1} = \mathcal{P}^1$. Similarly, the actions that can be played by any student $i \in I^h$ at node $h \in H^2$ are given by $M_i^h = \mathcal{P}^2$. Suppose now that $h \in H^k$, and let $h' = ((a_i)_{i \in I^h}, h) \in H^{k+1}$ be the node obtained from node h when each student $i \in I^h$ plays an action $a_i \in M_i^h$.

In the above, nothing has explicitly been stated about (i) the set of active students I^h at node $h \in H^2$ and (ii) the actions they can play at node $h \in H^3$. Both (i) and (ii) are determined by the **rules of the game**. As already explained in Section 3, these rules are given by the mechanisms ϕ^1 and ϕ^2 used in Rounds 1 and 2, respectively, and the correspondences ψ and γ. Here, the correspondence ψ determines the set of active students I^h at node $h = (a, h_1) \in H^2$. Note that the set of active students may depend on the Round 1 assignments that are determined by the mechanism $\phi^1(a)$. From Section 3, we know that the correspondence ψ prescribes that one of the following two cases prevail:

- $I^h = \{i \in I : \phi_i^1(a) = s_\emptyset\}$,
- $I^h(a) = I$.

The correspondence γ provides the actions that can be played by students in I^h at node $h \in H^3$, i.e., $\Pi_{i \in I^h} M_i^h$, where $M_i^h \subseteq \{\phi_i^1(\bar{a}), \phi_i^2(\hat{a})\}$ and $h = (\hat{a}, (\bar{a}, h_1))$. Moreover, the actions M_i^h that a student i can take at node $h = (\hat{a}, (\bar{a}, h_1)) \in H^3$ depends on the whether student i belongs to $I^{(\bar{a}, h_1)}$ or not. A more detailed description of the conditions (1) and (2a)–(2c) from Section 3 is as follows:

(1) if $i \notin I^{(\bar{a}, h_1)}$, then $M_i^h = \{\phi_i^1(\bar{a})\}$,

(2) if $i \in I^{(\bar{a}, h_1)}$, then one of the following three cases holds:

(a) $M_i^h = \{\phi_i^1(\bar{a}), \phi_i^2(\hat{a})\}$,

(b) $M_i^h = \begin{cases} \{\phi_i^1(\bar{a})\} & \text{if } \hat{a}_i = s_\emptyset \\ \{\phi_i^2(\hat{a})\} & \text{otherwise,} \end{cases}$

(c) $M_i^h = \begin{cases} \{\phi_i^1(\bar{a})\} & \text{if } \phi_i^2(\hat{a}) = s_\emptyset \\ \{\phi_i^2(\hat{a})\} & \text{otherwise.} \end{cases}$

Note that in case (2b), $\hat{a}_i = s_\emptyset$ is interpreted as not participating in Round 2 although student i is allowed to participate.

In the considered framework, a given problem $(S, I, q, P, \succ)$ and list of rules $(\phi^1, \phi^2, \psi, \gamma)$ induce the above described extensive form game and, consequently, the game $\Gamma = (I, H, M, \pi, P)$.

We end this appendix by stating a more formal definition of the notion of straightforwardness (see also Definition 1 in Section 3).

Definition 1. Given rules of the game $(\phi^1, \phi^2, \psi, \gamma)$, a student $i \in I$ plays a **straightforward** strategy m_i if:

> *Round* 1. For nodes $h \in H^1 = \{h_1\}$, the action m_i^h represents the true preferences of student i over the schools in $S^1 \cup \{s_\emptyset\}$.
>
> *Round* 2. For nodes $h = (\bar{a}, h_1) \in H^2$, the action m_i^h represents the true preferences of student i over the schools in $S^2 \cup \{s_\emptyset\}$ that are strictly preferred to $\phi_i^1(\bar{a})$ (note that $\phi_i^1(\bar{a})$ might be $\emptyset$).
>
> *Round* 3. For nodes $h \in H^3$, the action m_i^h for student i is always given by $m_i^h = \text{argmax}_{P_i} M_i^h$.

B Proofs

This appendix contains the proofs of all theoretical results presented in Sections 4 and 5.

Proof of Proposition 1. To prove the result, we provide counterexamples in which either S^1 or S^2 is empty.[26]

Case $S^1 = \emptyset$. Since ϕ^2 is individually irrational <wasteful> [unfair], there exists a problem such that ϕ^2 selects a matching which is individually irrational <wasteful> [unfair]. Suppose that $\phi^2(I, S^2, q^2, \succ^2, P^2)$ is individually irrational <wasteful> [unfair] and $S^1 = \emptyset$. Then, $(I, S, q, \succ, P) = (I, S^2, q^2, \succ^2, P^2)$ and for any (ψ, γ, ϕ^1) if there exists a straightforward SPNE the related equilibrium outcome, $\phi^2(I, S^2, q^2, \succ^2, P^2)$ is individually irrational <wasteful> [unfair].

Case $S^2 = \emptyset$. Since ϕ^1 is individually irrational <wasteful> [unfair], there exists a problem such that ϕ^1 selects a matching which is individually irrational <wasteful> [unfair]. Suppose that $\phi^1(I, S^1, q^1, \succ^1, P^1)$ is individually irrational <wasteful> [unfair] and $S^2 = \emptyset$. Then, $(I, S, q, \succ, P) = (I, S^1, q^1, \succ^1, P^1)$ and for any (ψ, γ, ϕ^2) if there exists a straightforward SPNE the related equilibrium outcome, $\phi^1(I, S^1, q^1, \succ^1, P^1)$ is individually irrational <wasteful> [unfair]. □

Proof of Proposition 2. The four parts of the proposition are proved one after another:

Part 1. Let $I = \{i\}$, $S^1 = \{s\}$, $S^2 = \{s'\}$, and $q = (1, 1)$. Suppose now that both schools regard student i as acceptable and that $s' P_i s P_i s_\emptyset$. Under the induced game, there exists a unique

[26] Note that one can easily modify these counterexamples by adding schools to these sets.

SPNE such that student i ranks $s_\emptyset$ above school s at the initial node h_1, student $i \in I^h$ ranks school s' over $s_\emptyset$ in any node $h \in H^2$, and student i selects the best option according to her true preferences in any node $h \in H^3$. This unique SPNE is not straightforward.

Part 2. Let $I = \{i, i'\}$, $S^1 = \{s\}$, $S^2 = \{s'\}$, $q = (1, 1)$, $i' \succ_s i \succ_s \emptyset$, $i' \succ_{s'} i \succ_{s'} \emptyset$, $s' P_i s P_i s_\emptyset$, and $s' P_{i'} s_\emptyset P_{i'} s$. Under the induced game, there exists a unique SPNE such that (i) student i ranks school s above $s_\emptyset$ and student i' ranks $s_\emptyset$ above school s at the initial node h_1, (ii) student i ranks $s_\emptyset$ over school s' (i.e. student i does not actively participate Round 2), and student i' ranks school s' above $s_\emptyset$ in any node $h \in H^2$, and (iii) students i and i' select the best option according to their true preferences in any node $h \in H^3$. Now, when both students play straightforward actions[27], student i is matched to $s_\emptyset$ and she can profitably deviate by ranking $s_\emptyset$ over s' in any node $h \in H^2$.

Part 3. It will be demonstrated that a straightforward SPNE induces a wasteful outcome for the schools in S^1. Let $I = \{i, i'\}$, $S^1 = \{s\}$, $S^2 = \{s'\}$, $q = (1, 1)$, $i \succ_s i' \succ_s \emptyset$, $i \succ_{s'} i' \succ_{s'} \emptyset$, $s' P_i s P_i s_\emptyset$, and $s P_{i'} s_\emptyset P_{i'} s'$. Consider now the following strategy profile: (i) student i ranks school s above $s_\emptyset$ at the initial node h_1, school s' above $s_\emptyset$ at any node $h \in H^2$, and accepts her most preferred school at any node $h \in H^3$, and (ii) student i' ranks school s above $s_\emptyset$ at the initial node h_1, school $s_\emptyset$ above school s' at any node $h \in H^2$, and accepts only school s whenever possible at any node $h \in H^3$. This strategy profile is a straightforward SPNE, and in the related outcome, student i is matched to school s' and student i' is matched to $s_\emptyset$. Hence, the seat at school s is wasted for student i'.

Part 4. It will be demonstrated that a straightforward SPNE induces a wasteful outcome for the schools in S^2. Let $I = \{i, i'\}$, $S^1 = \{s\}$, $S^2 = \{s'\}$, $q = (1, 1)$, $i \succ_s i' \succ_s \emptyset$, $i \succ_{s'} i' \succ_{s'} \emptyset$, $s P_i s' P_i s_\emptyset$ and $s' P_{i'} s_\emptyset P_{i'} s$. Consider now the following strategy profile: (i) student i ranks school s above $s_\emptyset$ at the initial node h_1, school s' above $s_\emptyset$ at any node $h \in H^2$, and accepts her most preferred school at any $h \in H^3$, (ii) student i' ranks $s_\emptyset$ above school s at the initial node h_1, school s' above $s_\emptyset$ at any node $h \in H^2$, and accepts only school s' whenever possible at any $h \in H^3$. This strategy profile is a straightforward SPNE, and in the related outcome, student i is matched to school s and student i' is matched to school $s_\emptyset$. Hence, the seat at school s' is wasted for student i'. □

Proof of Theorem 1. It is first established that there exists a straightforward SPNE. Let m be the straightforward strategy profile. Let also $h' \in H^2$ be the node that is reached under strategy profile m from the initial node h_1. To obtain a contradiction, suppose that student i can be matched to a more preferred school s according to her true preferences by deviating when all other students play straightforward strategies.

[27] That is, (i) student i ranks school s above $s_\emptyset$ and student i' ranks $s_\emptyset$ above school s at the initial node h_1, (ii) student i ranks school s' over $s_\emptyset$ and student i' ranks school s' above $s_\emptyset$ in any node $h \in H^2$, and (iii) students i and i' select the best option according to their true preferences in any node $h \in H^3$.

First note that, by definition of the rules $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$, each student in I has only one option to select in Round 3. Hence, by definition, each student has a dominant strategy in each node $h \in H^3$, which is the straightforward action.

Suppose now that school s belongs to S^1. Then, under strategy profile m, all seats of school s are occupied by students with higher priority than student i. Since the constrained serial dictatorship mechanism is used in Round 1,[28] student i's deviation will not affect the assignments of students with higher priority than i. That is, $s \notin S^1$.

By the above arguments, school s must belong to S^2. Let now $h'' \in H^2$ be the node reached from node h_1 when student i deviates and all other students play straightforward strategies. Then, at nodes h' and h'', students with higher priorities than student i play the same straightforward actions. Since the constrained serial dictatorship mechanism is also used in Round 2, student i's deviation will not affect the assignments of students with higher priority than i. That is, $s \notin S^2$. Hence, the strategy profile m is a SPNE.

Finally, it is demonstrated that any SPNE outcome is non-wasteful for the schools that are available in Round 2. To obtain a contradiction, suppose that μ is an outcome related to the SPNE (given actions m) but that there exists a school-student pair (s, i) such that $s \in S^2$, $|\mu_s| < q_s$ and $sP_i\mu_i$. Consider now the following strategy: student i plays the same actions in Round 1 and ranks school s as her top choice in Round 2 in any node $h \in H^2$. Since nothing changes in Round 1, we will end up at the same node in Round 2 as when student i plays according to m_i. Because the constrained serial dictatorship mechanism is used, school s will be available for student i and student i will then be assigned to school s as it is her top choice. Hence, any SPNE outcome is non-wasteful for the schools that participate in Round 2. □

Proof of Proposition 3. Consider an arbitrary grand problem $(S, I, q, P, \succ)$, let m be a SPNE strategy profile, and μ the induced equilibrium outcome.

It is first demonstrated that any SPNE is individually rational. To obtain a contradiction, suppose that μ is individually irrational. Then there exists a student i who prefers $s_\emptyset$ to her match μ_i. But then student i can be better off by ranking $s_\emptyset$ as her top choice at each node $h \in H^1 \cup H^2$ since she will be assigned to $s_\emptyset$ because an individually rational mechanism is used in both rounds. This contradicts that the strategy profile m is a SPNE.

It is next demonstrated that any SPNE is fair. We first focus on public schools. Suppose that there exist a student i and a public school s such that $sP_i\mu_i$ and $j \succ_s i$ for some $j \in \mu_s$. Then, at node h_1, student i can rank school s as her top choice and will, consequently, be assigned to school s. The latter conclusion follows from the fact that the constrained serial dictatorship mechanism is adopted (see the proof of Theorem 1). Consequently, whenever it is student i's turn, school s has an available seat and all remaining students have lower test scores than student i. This contradicts that m is a SPNE.

[28] See the discussion following Theorem 1.

Next consider private schools, and recall that under the old Turkish system, all students can participate in Round 2. Suppose now that there exist a student i and a private school s such that $sP_i\mu_i$ and $j \succ_s i$ for some $j \in \mu_s$. As explained in the above, student i can then be assigned to school s by submitting the same action at node h_1 and by ranking school s as her top choice in any node $h \in H^2$. This contradicts that m is a SPNE. □

Proof of Proposition 4. Consider an arbitrary grand problem $(S, I, q, P, \succ)$, let m be a SPNE strategy profile, and μ the induced equilibrium outcome.

It is first demonstrated that any SPNE is individually rational. To obtain a contradiction, suppose that μ is individually irrational. Then there exists a student i who prefers $s_\emptyset$ to her match μ_i. But then student i can be better off by ranking $s_\emptyset$ as her top choice at each node $h \in H^1 \cup H^2$ since she then will be assigned to $s_\emptyset$ because an individually rational mechanism is used in both rounds. This contradicts that the strategy profile m is a SPNE.

It is next demonstrated that any SPNE is non-wasteful and fair. We first focus on private schools. Suppose that there exist a student i and a private school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. In both cases, at node h_1, student i can rank school s as her top choice and will, consequently, be assigned to school s. The latter conclusion follows from the fact that the constrained serial dictatorship mechanism is adopted (see the proof of Theorem 1). Consequently, whenever it is student i's turn, school s has an available seat and all remaining students have lower test scores than student i. This contradicts that m is a SPNE.

Next consider public schools. Suppose first that there exist a student i who participates in Round 2 under strategy profile m and a public school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. As explained in the above, student i can be assigned to school s by submitting the same action at node h_1 and by ranking school s as top choice in any node $h \in H^2$. Suppose instead that there does not exist a student i who participates in Round 2 under strategy profile m and a public school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. Instead, there exist a student i who does not participate in Round 2 under strategy profile m and a public school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. Consider now the strategy m'_i in which student i ranks $s_\emptyset$ as her top choice at node h_1, and school s as her top choice in any node $h \in H^2$. Let $m' = (m'_i, m_{-i})$. Under strategy profile m' student i is unassigned in Round 1 and she can, therefore, participate in Round 2. Moreover, the set of students participating in Round 2 under strategy profile m' is a subset of students participating in Round 2 under strategy-profile m. Then, since the constrained serial dictatorship mechanism is adopted in Round 2, in any node $h \in H^2$, school s will have an available seat when it is student i's turn. That contradicts that m is a SPNE. □

Proof of Proposition 5. Consider an arbitrary grand problem $(S, I, q, P, \succ)$, let m be a SPNE strategy profile, and μ the induced equilibrium outcome.

It is first demonstrated that any SPNE is individually rational. To obtain a contradiction, suppose that μ is individually irrational. Then there exists a student i who prefers $s_\emptyset$ to her match

μ_i. But then student i can be better off by ranking $s_\emptyset$ as her top choice at each node $h \in H^1 \cup H^2$ since she then will be assigned to $s_\emptyset$ because an individually rational mechanism is used in both rounds. This contradicts that the strategy profile m is a SPNE.

It is next demonstrated that any SPNE is non-wasteful and fair for private schools. Suppose that there exist a student i and a private school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. In both cases, at node h_1, student i can rank school s as her top choice and will, consequently, be assigned to school s. The latter conclusion follows from the fact that in both cases, student i will be among the top q_s students applying to school s according to $\succ_s$. This contradicts that m is a SPNE.

Finally, we focus on the public schools. Suppose that there exist a student i and public school s such that $sP_i\mu_i$ and $|\mu_s| < q_s$. We first consider the case such that student i is unassigned in Round 1 and participates in Round 2 under strategy m. In this case, student i can get school s by only changing her ranking in Round 2 by ranking s at the top for any $h \in H^2$. Call this strategy m_i' and let $m' = (m_i', m_{-i})$. Since the rankings submitted in Round 1 are unchanged, m will lead to the same subgame in Round 1. By using the sequential version of the deferred acceptance algorithm (McVitie and Wilson, 1971), one can easily see that under strategy profile m' i is assigned to s.

Now consider the case where student i is assigned in Round 1. Recall that in order to participate in Round 2, i needs to be unassigned in Round 1. Then, consider the following strategy m_i' for i: rank $s_\emptyset$ over all schools at h_1 and rank s over all schools at any node $h \in H^2$. Let $m' = (m_i', m_{-i})$. Under strategy m' let $h' \in H^2$ be the node reached in Round 2. Then, the set of students active in node h' will be a subset of the active students under m union i. If we restrict them to playing their weakly undominated strategy, i.e., their true preferences over the public schools then, by using the sequential deferred acceptance algorithm, and the fact that the deferred acceptance algorithm is population monotonic we can show that when it is student i's turn she can get school s. □

Proposition 6. Suppose that all schools in S have the same relative priorities over acceptable students. Let $(SD, SD, \psi^*, \gamma^*)$ be the rules of the admission system. Suppose all students play straightforward actions. For any problem $((S^{pu}, S^{pr}), I, q, P, \succ)$, let μ and ν be the outcomes of this system when private school $s \in S^{pr}$ participates in the first round and the second round while keeping everything else the same, respectively. Then, $\nu_s \succsim_s \mu_s$.

Proof. In the rest of the proof, we call the case in which all private schools participate in Round 1 as Case 1, and the case in which only private school s participates in Round 2 as Case 2. Let μ^1 and ν^1 be the matchings selected in the first round of Case 1 and Case 2, respectively. When all students and schools play straightforward actions, the outcome in the first round is equivalent to the outcome of the serial dictatorship mechanism under true preferences over the available schools in the first round. It is easy to see that when the number of available schools decreases, all students become (weakly) worse off under the serial dictatorship mechanism. Hence, under

Case 2, all students in μ_s^1 are assigned to a worse private school than s in ν^1. All the other students become weakly worse off under ν^1.

On the contrary, suppose $\mu_s \succ_s \nu_s$. Then, there exists at least one student $\bar{i}$ such that $\mu_{\bar{i}} = s$, $\nu_{\bar{i}} \neq s$ and $\nu_{\bar{i}} P_i \mu_{\bar{i}}$. To see this, we consider the following cases in which our claim does not hold: **Case (a)** $\nu_i = s$ for each $i \in \mu_s$, and **Case (b)** $\mu_s \neq \nu_s$ and $\mu_i R_i \nu_i$ for all $i \in \mu_s$. If Case (a) is true, then $\mu_s \subseteq \nu_s$. Since all students in ν_s are acceptable for s, $\nu_s \succsim_s \mu_s$. If Case (b) is true, then under ν school s fills all its available seats, i.e., $|\nu_s| = q_s$, and there exists at least one student $i \in \mu_s$ such that $\mu_i P_i \nu_i$. Without loss of generality, let i have the highest priority among such students. That is, if $j \in \mu_s$ and $j \succ_s i$, then $\nu_j = s$. Moreover, all students in ν_s have higher priority than i. Therefore, $\nu_s \succ_s \mu_s$.

Without loss of generality suppose i is the student with highest priority such that $\mu_i = s$ but $\nu_i \neq s$ and $\nu_i P_i s$. Let $s' = \nu_i$. Since s is the only private school participating in Round 2 under Case 2, s' needs to be a public school. Since $\mu_i R_i \mu_i^1$ and $s' P_i \mu_i$, under Case 1 student i also applies to s' in Round 2. Since i is not assigned to s' in Case 1 all assignees of s' have higher priority than i and $|\mu_{s'}| = q_{s'}$. Then, there exists at least one student who is assigned to s' in μ but not in ν. Let $j \in (\mu_{s'} \setminus \nu_{s'})$. Then, j has higher priority than i and $s' P_j s$. Then j has to be assigned to a better public school than s' under ν. If we apply the same reasoning for student j, then due to the finite number of students and schools we will have a contradiction ultimately. That is, there does not exist $i \in \mu_s$ but $i \notin \nu_s$ and $\nu_i P_i s$. Hence if $i \in \mu_s$ but $i \notin \nu_s$ then $s P_i \nu_s$. That is, all slots of s are filled with students who have higher priority than i. Which means that a better set of students is assigned to s under Case 2 and $\nu_s \succsim_s \mu_s$. □

C Decentralized Turkish Private School Admission Game

Because almost all private schools in Turkey have the same ranking over the students, it is assumed, without loss of generality, that for all schools $s, s' \in S^{pr}$ and all students $i, i' \in I$, it holds that $i \succ_s i'$ if and only if $i \succ_{s'} i'$. The priority order of the schools are determined based on students' Grade Point Averages and schools' cut-off scores. Let $t = (t_i)_{i \in I}$ be the **test score** profile where $t_i \in \mathbb{R}_{++}$ is the test score of student $i \in I$. We assume that for all $i, i' \in I$, $t_i \neq t_{i'}$. Furthermore, for any school $s \in S$, $i \succ_s i'$ if and only if $t_i > t_{i'}$. Let $x = (x_s)_{s \in S}$ be the **cut-off score** vector where x_s is the cut-off score of school s within the possible discrete score set of $X \subseteq \mathbb{R}_+$. For any school $s \in S$ and any student $i \in I$, $i \succ_s \emptyset$ if and only if $t_i \geq x_s$. Because private schools may have different cut-off scores, they might also have different sets of acceptable students.

The admission for the private schools are done through decentralized system which can be formalized as a sequential game (see also Section 5). The players in this sequential game are the students and the private schools. Throughout the game, schools and students move sequentially. For the sake of clarity, we group these consecutive steps together and we call it a round. In each Round k:

- Each private school s announces a cut-off score $x_s^k \in X = \{0, x_1, x_2, ..., x_{max}\}$ where X is a countably finite set of test scores and x_{max} is the maximum possible score that a student can take in the centralized exam. We assume that schools are required to weakly reduce their announced cut-off scores compared to x_s^{k-1}, i.e., $x_s^k \leq x_s^{k-1}$.

- After observing the cut-off scores, each student i chooses a subset of schools A_i^k among the ones she has not rejected before, possibly the empty set, where $t_i \geq x_s^k$ for each $s \in A_i^k$.

- Given the applicants in this round, each school s offers acceptance to the top $\min\{q_s - a_s^k, b_s^k\}$ acceptable applicants according to x_s^k where b_s^k is the number of applicants in round t and a_s^k is the number of students holding school s's offer in Round $k-1$ and $a_s^1 = 0$.

- Each student s holds at most one offer among the ones she has received in Round k and the school seat that she may hold from a previous round and rejects the rest.

The game ends when students do not reject any more offers. Since the number of schools and students are finite, this game terminates. Each student is assigned to the school whose offer she has been holding when the game terminates. In the game described above, each school gives offers automatically according to its preference order over the students. Hence, a school s can be strategic only in the very first step of each round when the cut-off score is set. In Theorem 2 below, we show that this admission game has a unique SPNE outcome which is equivalent to the unique stable matching under true preferences and priorities implied by the test scores and the true cut-off scores. This unique stable matching can be described as a **constrained serial dictatorship** mechanism.[29]

Theorem 2. For given I, S, t and q, the decentralized admission game for the Turkish private schools has a unique SPNE outcome which is equivalent to the constrained serial dictatorship outcome under true preferences and true cut-off scores.

Proof. Let μ be the outcome of the constrained serial dictatorship mechanism under true preferences and true cut-off scores. One can easily verify that μ is the unique stable matching under true preferences and true cut-off scores. It will be demonstrated that there is a unique SPNE outcome of the game and it is μ. The result is proved by induction. Let student i_m be the student with the m^{th} highest test score for the private schools.

[29]Unlike the standard serial dictatorship mechanism, here students are allowed to choose schools which consider them acceptable. The constrained serial dictatorship mechanism is defined through a priority ordering induced by the test scores. Accordingly, a priority ordering is a one-to-one and onto function $f : \{1, ..., |I|\} \to I$ where $f(k)$ is the student with k^{th} highest test score. The constrained serial dictatorship is defined iteratively as follows:

Step 1. The student with the highest test score $f(1)$ selects her most preferred school among the schools which consider her acceptable, i.e., with $t_{f(1)} \geq x_s$.

Step k>1. The student with the k^{th} highest test score $f(k)$ selects her most preferred school among the remaining schools which consider her acceptable, i.e., with $t_{f(k)} \geq x_s$.

It is first proved that in any SPNE outcome, student i_1 is assigned to school μ_{i_1}. Let S_1 be set of schools which student i_1 considers acceptable, i.e., $sP_1s_\emptyset$ for all $s \in S_1$. Note that $\mu_{i_1} \in S_1 \cup s_\emptyset$. We need to consider two cases:

- *Case* $\mu_{i_1} = s_\emptyset$. In this case, student i_1's test score is lower than the true cut-off score of the schools in S_1. Since student i_1 has the highest score, any school $s \in S_1$ considers all students as unacceptable. Hence, in any SPNE outcome, these schools will not admit any student. Otherwise, each school in S_1 can deviate by setting the cut-off score strictly higher than t_{i_1}. Then, if student i_1 is assigned to a school in a SPNE outcome, then that school is unacceptable for student i_1. However, by not applying to any school in any subgame leaves student i_1 unassigned and therefore, student i_1 is assigned to school $s_\emptyset$ in any SPNE outcome.

- *Case* $\mu_{i_1} \in S_1$. Consider any subgame in which, in some step, school μ_{i_1} sets its cut-off score less than or equal to t_{i_1}. Then, student i_1 can get an offer from school μ_{i_1} whenever she applies to it. Hence, in any equilibrium of the subgame following the node in which school μ_{i_1} sets its cut-off score less than or equal to t_{i_1}, student i_1 will not be assigned to a less preferred school than than μ_{i_1}. Moreover, if there exists an equilibrium in which student i_1 is assigned to a more preferred school than μ_{i_1}, then the school she is assigned to considers her and all other students as unacceptable. Therefore, that school will profitably deviate by setting its cut-off strictly higher than t_{i_1} in every subgame. Thus, in that subgame, student i_1 will be assigned to school μ_{i_1}. Moreover, there cannot be any SPNE in which school μ_{i_1} reports the cut-off score higher than t_{i_1}. Otherwise, setting the cut-off score equal to t_{i_1} in the very first step would be a profitable deviation. Hence, in any SPNE outcome, student i_1 is assigned to school μ_{i_1}.

The above two cases complete the proof for the case when $k = 1$. We can now use an induction argument for $k > 1$. For this purpose, suppose that, in any SPNE, student $i_{k'}$ is assigned to school $\mu_{i_{k'}}$ for all $k' < k$. Now we need to prove that, in any SPNE outcome, student i_k is assigned to school μ_{i_k}. Let S_k be set of schools which student i_k considers acceptable, i.e., $sP_ks_\emptyset$ for all $s \in S_k$. Note that $\mu_{i_k} \in S_k \cup s_\emptyset$. We need to consider two cases:

- *Case* $\mu_{i_k} = s_\emptyset$. Because student i_k is unassigned under μ either student i_k's test score is lower than the true cut-off of the schools in S_k, or those schools have already filled their capacity, i.e, for each school $s \in S_k$ there exists a subset $\bar{I} \subseteq \{i_1, i_2, ..., i_{k-1}\}$ such that $\mu_j = s$ for all $j \in \bar{I}$ and $q_s = |\bar{I}|$. If the former case is true and student i_k is assigned to some school $s \in S_k$, then school s has a profitable deviation by setting its cut-off strictly higher than t_{i_k}. If the latter case holds for some school $s \in S_k$, then, by feasibility, student i_k cannot be assigned to school s in any SPNE outcome. Moreover, student i_k cannot be assigned to any school $s \in S \backslash S_k$ as she then would deviate by not applying to any school in any round making sure that she would be unassigned, i.e., $s_\emptyset P_k s$ for all $s \in S \backslash S_k$.

- *Case* $\mu_{i_k} \in S_k$. Consider any subgame in which, in some step, school μ_{i_k} sets its cut-off score less than or equal to t_{i_k}. Then, student i_k can get an offer from school μ_{i_k} whenever she applies to it. Hence, in any equilibrium of the subgame following the node in which school μ_{i_k} sets its cut-off score less than or equal to t_{i_k}, student i_k will not be assigned to a school which is less preferred than school μ_{i_k}. Moreover, if there exists an equilibrium in which student i_k is assigned to a more preferred school than μ_{i_k}, then the school she is assigned to considers her and all students with test scores less than t_{i_k} as unacceptable. Therefore, that school will profitably deviate by setting its cut-off score higher than t_{i_k} in every subgame. Hence, in that subgame, student i_k will be assigned to school μ_{i_k}. Moreover, there cannot be any SPNE in which school μ_{i_k} reports a cut-off score higher than t_{i_k}. Otherwise, some seats at school μ_{i_k} will be wasted and setting a cut-off equal to t_{i_k} in the very first step would be a profitable deviation. Hence, in any SPNE outcome, student i_k is assigned to school μ_{i_k}.

The above two cases complete the proof for an arbitrary $k > 1$ and, consequently, the entire proof. □

Finally, we consider a version of the decentralized admission game in which private schools report their cut-off score only once and at the beginning of the game. Under this setting, we get the same equilibrium outcome as in the decentralized game described in the above.

Corollary 4. The decentralized admission game for the private schools in which private schools report cut-off scores only once has a unique SPNE outcome which is equivalent to the constrained serial dictatorship outcome under true preferences and true cut-off scores.

Note that the proof of Theorem 2 does not rely on the fact that the private schools report cut-off scores more than once. Hence, the proof follows directly follows from the proof of Theorem 2.

Finally note that Theorem 2 and Corollary 4 show that the unique equilibrium outcome does not depend on the fact that private schools report cut-off scores just in the beginning of the mechanism or for every subgame.

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